Mathematical Induction

Proof by mathematical induction

A commonly-used trick in mathematical proofs is "**proof by mathematical induction**". It is mainly used for proving that a statement involving n $\in \mathbb{N}$ is true for all n.

<u>Example</u>: Notice the following pattern:

$$1 = 1^2$$
$$1 + 3 = 2^2$$
$$1 + 3 + 5 = 3^2$$
$$1 + 3 + 5 + 7 = 4^2$$
$$\ldots\ldots$$

We guess that the general pattern holds i.e.

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \text{ for any } n \in \mathbb{N}.$$

How to prove it? We can use mathematical induction. The procedure is as follows:

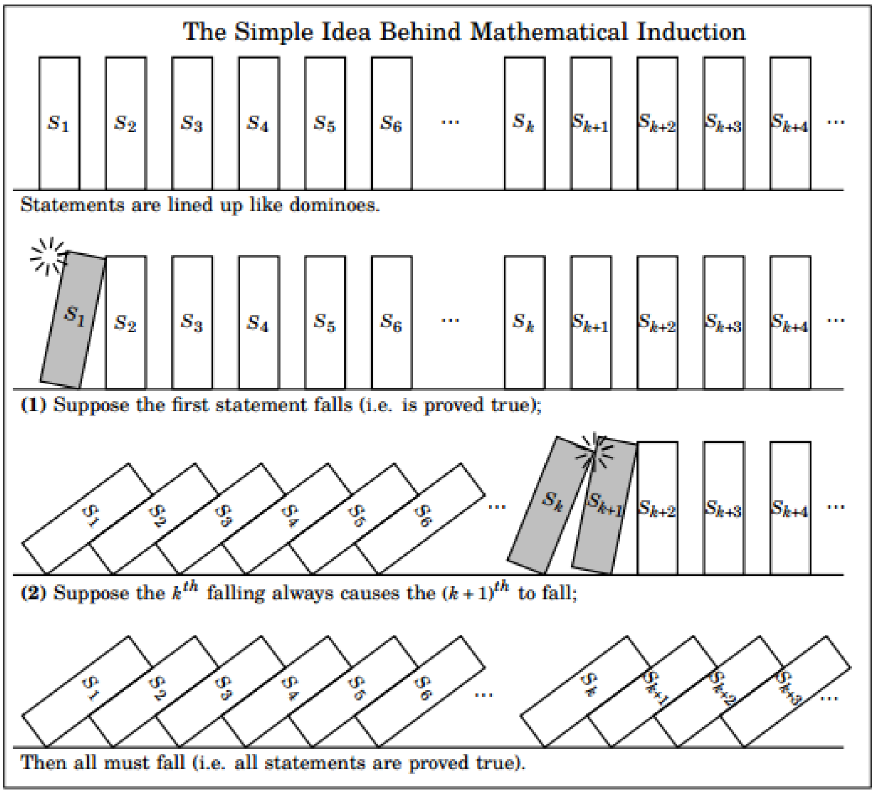
Prove it is true for n = 1: $1 = 1^2$

Assume it is true for n = k $\in \mathbb{N}$: $1 + 3 + 5 + \cdots + (2k - 1) = k^2$

Need to show that it is true for n = k + 1:

$$1 + 3 + 5 + \cdots + (2(k + 1) - 1) = (k + 1)^2$$

The following diagram explains why mathematical induction works:



<u>Exercise</u>:

Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(2n+1)(n+1)}{6}$ for any n $\in \mathbb{N}$.

Prove it is true for n = 1:

Assume it is true for n = k $\in \mathbb{N}$:

Need to show that it is true for n = k + 1:

Functions

What is a function?

Definition: Let A and B be sets. A **function** f is a "rule" which assigns one object in B to each of the objects in A. In this case, we call A to be the **domain** of f and B to be the **range** of f. We denote all the information collectively by the symbol

$$f: A \to B$$

For any $x \in A$, we let $f(x)$ be the object in B assigned by f i.e. $f(x) \in B$.

Examples of functions:

1. Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = 0$ for any $x \in \mathbb{R}$. f is called the **zero function**. The domain and range of f are $\mathbb{R}$. More generally, if we define $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) = k$ for any $x \in \mathbb{R}$, where k is a fixed real number, then f is called a **constant function**.
2. Let $g: \mathbb{R} \to \mathbb{R}$ be a function such that $g(x) = x^2$ for all $x \in \mathbb{R}$. For example, $g(1) = 1^2 = 1,\ g(2) = 2^2 = 4$.
3. Let P be the set of all people in the world. We define the function $m: P \to P$ such that $m(p) =$ biological mother of p, for any $p \in P$.
4. Let T be the set of all triangles in the plane. We define the function $a: T \to \mathbb{R}$ such that $a(t) =$ the area of triangle t, for any $t \in T$. For example, if t is a right-angled triangle with sides 3, 4 and 5, then $a(t) = 6$.
5. Let $h: \mathbb{N} \to \mathbb{Z}$ be a function such that $h(n) = \begin{cases} n, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd} \end{cases}$, for any $n \in \mathbb{N}$.

   For example, $h(1) = 1 - 1 = 0,\ h(2) = 2,\ h(3) = 3 - 1 = 2$.

Examples of non-functions:

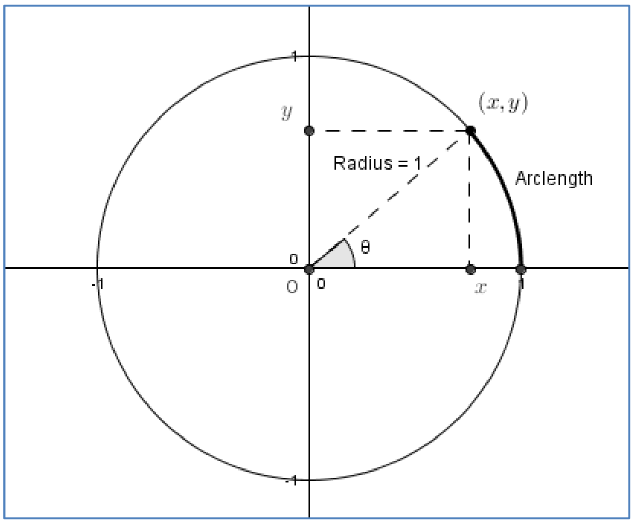
1. Let T be the set of all triangles in the plane. We define $b: (0, \infty) \to T$ such that $b(x) =$ a triangle t whose area is x, for any $t \in (0, \infty)$. b is NOT a function because more than one triangle has the given area.
2. We define $s: \mathbb{R} \to \mathbb{R}$ such that $s(x) = \sqrt{x}$, for any $x \in \mathbb{R}$. s is NOT a function because $\sqrt{x}$ is not a well-defined real number when $x < 0$.

Remark: Various disciplines of mathematics correspond to the studies of different types of functions.

Trigonometric Functions

Unit circle in $\mathbb{R}^2$

<u>Definition</u>: Unit circle is a circle with radius 1 centered at the origin in $\mathbb{R}^2$.



Suppose (x, y) is the coordinates of any point on the unit circle. By Pythagoras Theorem, we have

$$x^2 + y^2 = 1.$$

(This equation holds even for negative values of x and y.)

Hence, the unit circle can be regarded as

$$\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}.$$

Radian of an angle

For any point on the unit circle, we can measure angle θ, as shown in the above figure, from the x-axis to the line joining the point and origin in anti-clockwise direction.

When measuring the size of an angle, we usually use "degree" (e.g. $90°$ for right angle). However, for more advanced mathematics, another unit for angle, called "**radian**", is more commonly used.

<u>Definition</u>: **Angle in radian** is the arclength of the circular arc on the unit circle that subtends the angle.

<u>Examples</u>: $360° = 2\pi \text{ rad}$, $180° = \pi \text{ rad}$, $90° = \frac{\pi}{2} \text{ rad}$.

<u>Remark</u>: The name of the unit "rad" is often omitted.

<u>Theorem</u>: Angle in degree = angle in radian $\times \frac{180°}{\pi}$.

<u>Proof</u>: By definition, $1\text{ rad} = \frac{180°}{\pi}$. So the result follows by proportion.

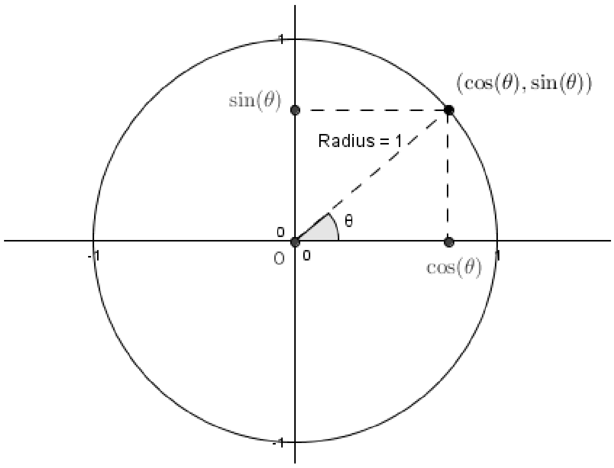
So far we assume that $\theta \in [0,2\pi]$. However, we can define angles greater than 2π or smaller than 0 as follows:

Consider $\theta = \frac{7\pi}{3}$. Imagine a point P travels on the unit circle by distance $\frac{7\pi}{3}$ from $(1,0)$ in **anticlockwise** direction. Then θ is the angle between the x-axis and the line joining the origin and the point P in its final position i.e. θ is the same as $\frac{\pi}{3}$.

For negative angle $\theta = -\frac{\pi}{3}$, the definition is similar except that now the point P travels in **clockwise** direction i.e. θ is the same as $\frac{5\pi}{3}$.

Sine and cosine functions

Now we define two very important functions:



<u>Definition</u>: $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ functions such that

$$f(\theta) = x$$
$$g(\theta) = y$$

for any $\theta \in \mathbb{R}$, where (x, y) is the point defined on the unit circle as shown in the figure. g is called the **sine function** and f is called the **cosine function**.

<u>Notation</u>: $g(\theta) = \sin(\theta)$ and $f(\theta) = \cos(\theta)$.

<u>Exercise</u>: Fill the following table:

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$
$\sin(\theta)$								
$\cos(\theta)$								