EPGL Program M801 – Problem-solving

Exercise 1

Q1.1 In ΔABC, given that $2B = A + C$,

prove that $(a+b+c)(a-b+c)=3ac$.

Q1.2 Given that $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0$;

Prove that $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

Q1.3 Prove that, for any integers x and y, we always have

$$\frac{1}{x} + \frac{1}{y} \neq \frac{1}{x+y} .$$

Q1.4 Prove that, given a positive odd integer n, there exist integers x and y, such that $5x^2 + 11y^2 - 1$ is a multiple of n.

Q1.5 Let a, b and c be the sides of a triangle, and $m > 0$,

prove that $\frac{a}{a+m} + \frac{b}{b+m} > \frac{c}{c+m}$.

Q1.6 Prove that there exist n positive integers $x_1, x_2, x_3, \dots . x_n$, such that $x_1 + x_2 + x_3 + \cdots + x_n = x_1 \cdot x_2 \cdot x_3 \dots \cdot x_n$ for $n \geq 2$.

Q1.7 Let N(n) denotes the number of distinct factors of the positive integer n. For example, N(24) = 8, since there are 8 distinct factors of 24, namely 1 , 2 , 3, 4, 6, 8, 12, 24. Determine whether N(1) + N(2) + ... +N(1993) is odd or even.

Q1.8 Given that $a, b, c, d \in \mathbb{N}$ and $a^4 + b^4 + c^4 + d^4 = 4abcd$, prove that a = b = c = d .

Q1.9 Find all ordered pairs (m,n) where m, n $\in \mathbb{N}$, such that $\frac{n^3+1}{mn-1}$ is an integer.

Q1.10 The sum of two natural numbers x and y is divisible by 7, prove that $x^7 + y^7$ is divisible by 49.