Secondary School-HKUST Dual Program
Pre-stage Level Physics

Lecture 11
Applications of Integration II

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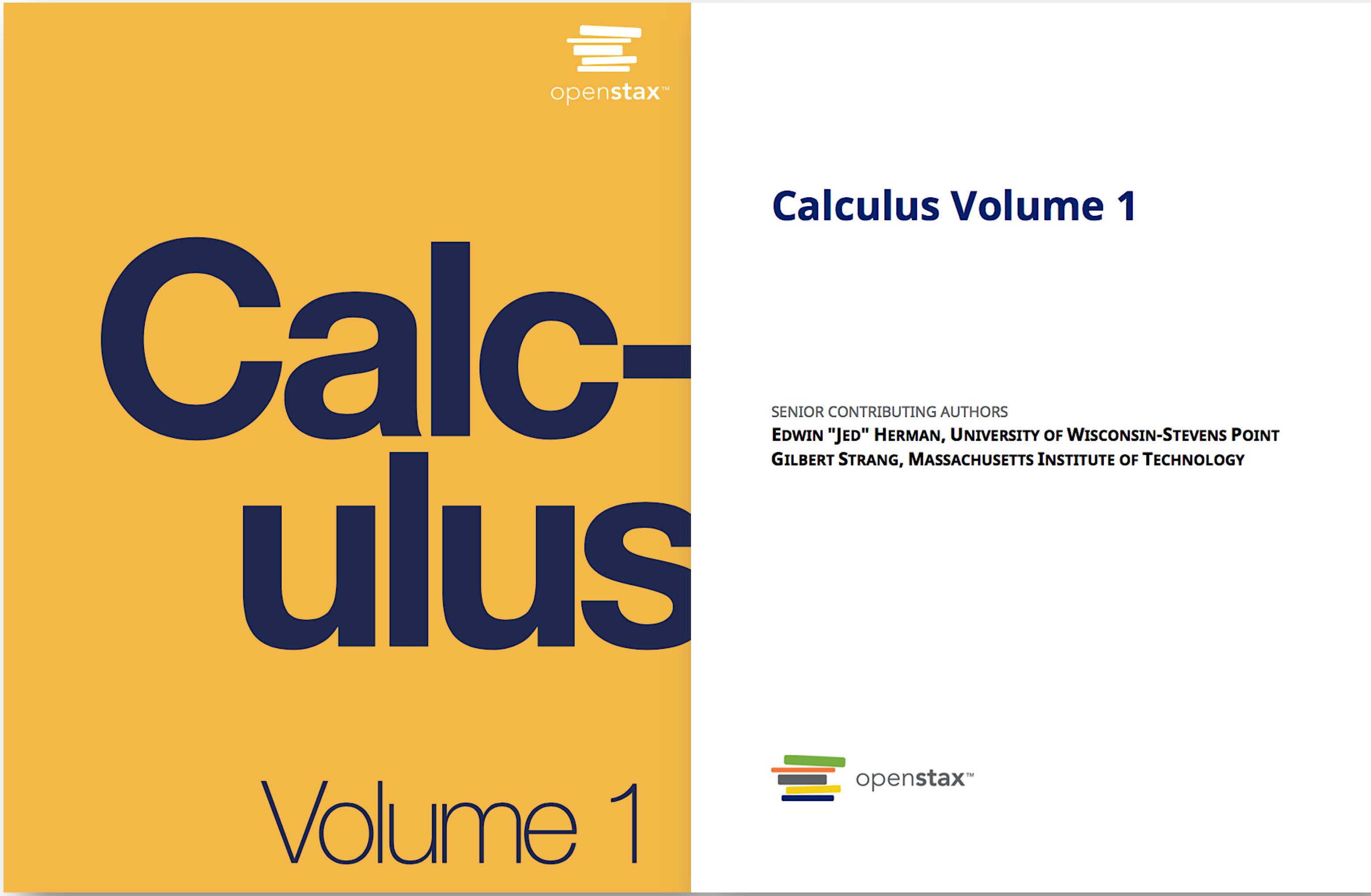
✧ These lecture notes are excerpts from Chapter 6 of the following free online textbook:

https://openstax.org/details/books/calculus-volume-1



(6.5) Physical Applications

Mass and Density

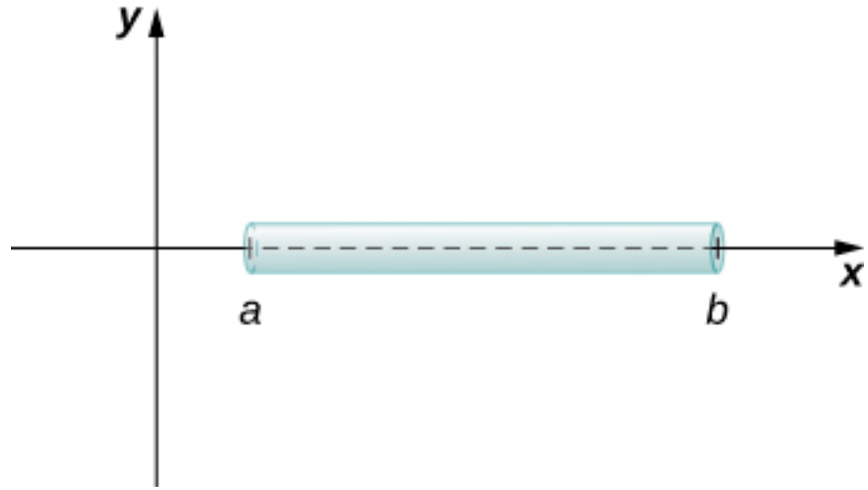


Figure 6.48 We can calculate the mass of a thin rod oriented along the x-axis by integrating its density function.

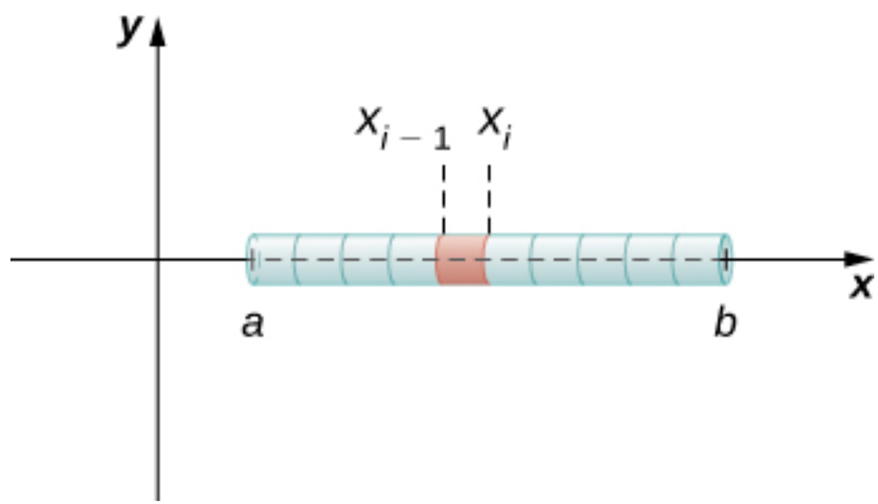


Figure 6.49 A representative segment of the rod.

(6.5) Physical Applications

The mass m_i of the segment of the rod from x_{i-1} to x_i is approximated by

$$m_i \approx \rho(x_i^*)(x_i - x_{i-1}) = \rho(x_i^*)\Delta x.$$

Adding the masses of all the segments gives us an approximation for the mass of the entire rod:

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} \rho(x_i^*)\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we get an expression for the exact mass of the rod:

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(x_i^*)\Delta x = \int_a^b \rho(x)dx.$$

(6.5) Physical Applications

Theorem 6.7: Mass–Density Formula of a One-Dimensional Object

Given a thin rod oriented along the x-axis over the interval $[a, b]$, let $\rho(x)$ denote a linear density function giving the density of the rod at a point x in the interval. Then the mass of the rod is given by

$$m = \int_a^b \rho(x)dx. \tag{6.10}$$

(6.5) Physical Applications

Example 6.23

Calculating Mass from Linear Density

Consider a thin rod oriented on the x-axis over the interval $[\pi/2, \pi]$. If the density of the rod is given by $\rho(x) = \sin x$, what is the mass of the rod?

Solution

Applying **Equation 6.10** directly, we have

$$m = \int_a^b \rho(x)dx = \int_{\pi/2}^{\pi} \sin x\, dx = -\cos x\Big|_{\pi/2}^{\pi} = 1.$$

(6.5) Physical Applications

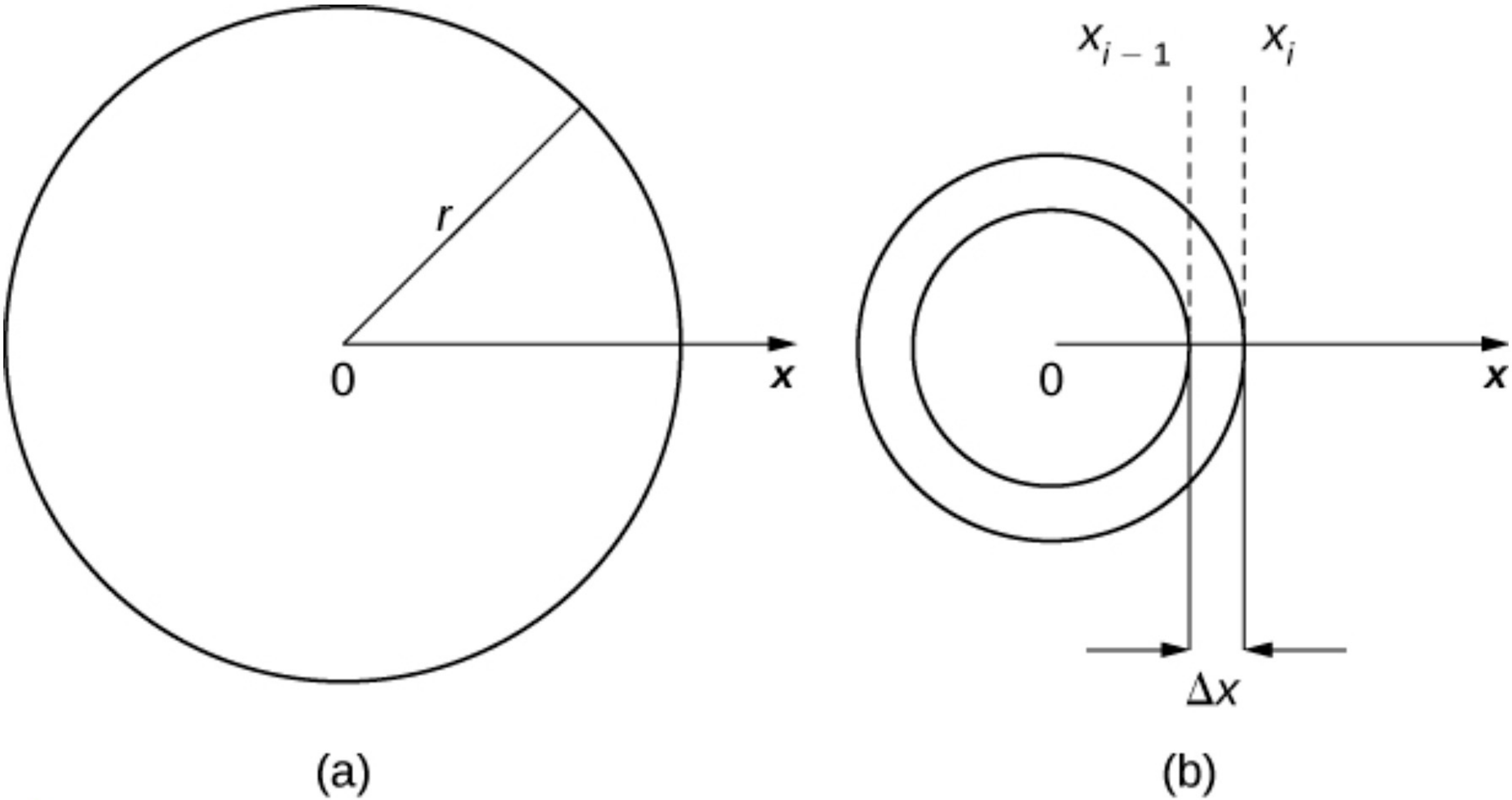


Figure 6.50 (a) A thin disk in the *xy*-plane. (b) A representative washer.

(6.5) Physical Applications

We now approximate the density and area of the washer to calculate an approximate mass, m_i. Note that the area of the washer is given by

$$\begin{aligned} A_i &= \pi(x_i)^2 - \pi(x_{i-1})^2 \\ &= \pi\left[x_i^2 - x_{i-1}^2\right] \\ &= \pi(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \pi(x_i + x_{i-1})\Delta x. \end{aligned}$$

You may recall that we had an expression similar to this when we were computing volumes by shells. As we did there, we use $x_i^* \approx (x_i + x_{i-1})/2$ to approximate the average radius of the washer. We obtain

$$A_i = \pi(x_i + x_{i-1})\Delta x \approx 2\pi x_i^* \Delta x.$$

(6.5) Physical Applications

Using $\rho(x_i^*)$ to approximate the density of the washer, we approximate the mass of the washer by

$$m_i \approx 2\pi x_i^* \rho(x_i^*)\Delta x.$$

Adding up the masses of the washers, we see the mass m of the entire disk is approximated by

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x.$$

We again recognize this as a Riemann sum, and take the limit as $n \to \infty$. This gives us

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x = \int_0^r 2\pi x \rho(x) dx.$$

(6.5) Physical Applications

Theorem 6.8: Mass–Density Formula of a Circular Object

Let $\rho(x)$ be an integrable function representing the radial density of a disk of radius r. Then the mass of the disk is given by

$$m = \int_0^r 2\pi x \rho(x) dx. \quad (6.11)$$

(6.5) Physical Applications

Example 6.24

Calculating Mass from Radial Density

Let $\rho(x) = \sqrt{x}$ represent the radial density of a disk. Calculate the mass of a disk of radius 4.

Solution

Applying the formula, we find

$$\begin{aligned} m &= \int_0^r 2\pi x \rho(x)dx \\ &= \int_0^4 2\pi x\sqrt{x}dx = 2\pi \int_0^4 x^{3/2}dx \\ &= 2\pi \frac{2}{5} x^{5/2}\Big|_0^4 = \frac{4\pi}{5}[32] = \frac{128\pi}{5}. \end{aligned}$$

(6.5) Physical Applications

Work Done by a Force

Suppose we have a variable force $F(x)$ that moves an object in a positive direction along the x-axis from point a to point b. To calculate the work done, we partition the interval $[a, b]$ and estimate the work done over each subinterval. So, for $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of the interval $[a, b],$ and for $i = 1, 2,\ldots, n,$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. To calculate the work done to move an object from point x_{i-1} to point $x_i,$ we assume the force is roughly constant over the interval, and use $F(x_i^*)$ to approximate the force. The work done over the interval $[x_{i-1}, x_i],$ then, is given by

$$W_i \approx F(x_i^*)(x_i - x_{i-1}) = F(x_i^*)\Delta x.$$

(6.5) Physical Applications

$$W_i \approx F(x_i^*)(x_i - x_{i-1}) = F(x_i^*)\Delta x.$$

Therefore, the work done over the interval $[a, b]$ is approximately

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(x_i^*)\Delta x.$$

Taking the limit of this expression as $n \to \infty$ gives us the exact value for work:

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} F(x_i^*)\Delta x = \int_a^b F(x)dx.$$

(6.5) Physical Applications

Definition

If a variable force $F(x)$ moves an object in a positive direction along the x-axis from point a to point b, then the **work** done on the object is

$$W = \int_a^b F(x)dx. \tag{6.12}$$

Note that if F is constant, the integral evaluates to $F \cdot (b - a) = F \cdot d,$ which is the formula we stated at the beginning of this section

(6.5) Physical Applications

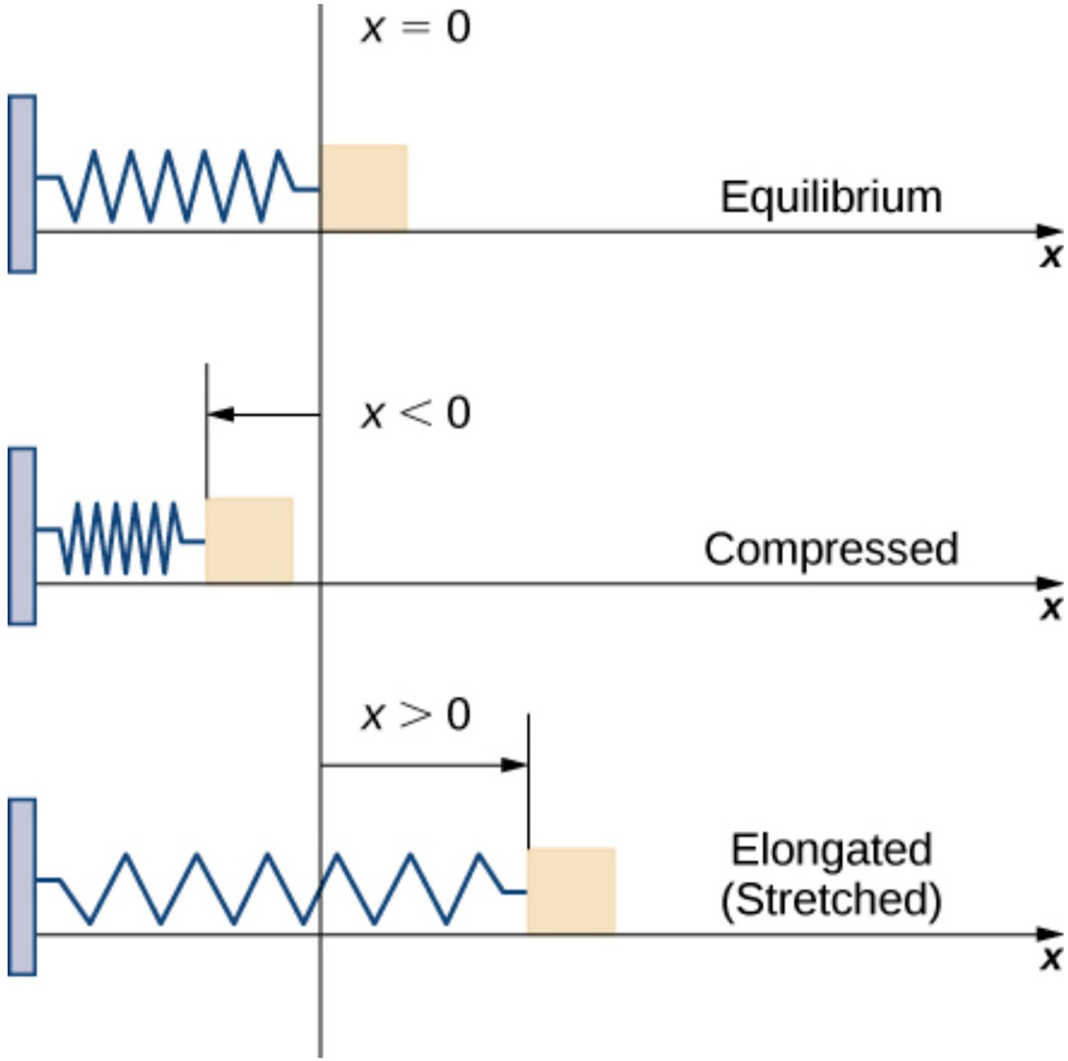


Figure 6.51 A block attached to a horizontal spring at equilibrium, compressed, and elongated.

(6.5) Physical Applications

According to **Hooke's law**, the force required to compress or stretch a spring from an equilibrium position is given by $F(x) = kx$, for some constant k. The value of k depends on the physical characteristics of the spring. The constant k is called the *spring constant* and is always positive. We can use this information to calculate the work done to compress or elongate a spring, as shown in the following example.

Example 6.25

The Work Required to Stretch or Compress a Spring

Suppose it takes a force of 10 N (in the negative direction) to compress a spring 0.2 m from the equilibrium position. How much work is done to stretch the spring 0.5 m from the equilibrium position?

Solution

First find the spring constant, k. When $x = -0.2$, we know $F(x) = -10$, so

$$\begin{aligned} F(x) &= kx \\ -10 &= k(-0.2) \\ k &= 50 \end{aligned}$$

and $F(x) = 50x$. Then, to calculate work, we integrate the force function, obtaining

$$W = \int_a^b F(x)dx = \int_0^{0.5} 50x\,dx = 25x^2\Big|_0^{0.5} = 6.25.$$

The work done to stretch the spring is 6.25 J.

(6.5) Physical Applications

Work Done in Pumping

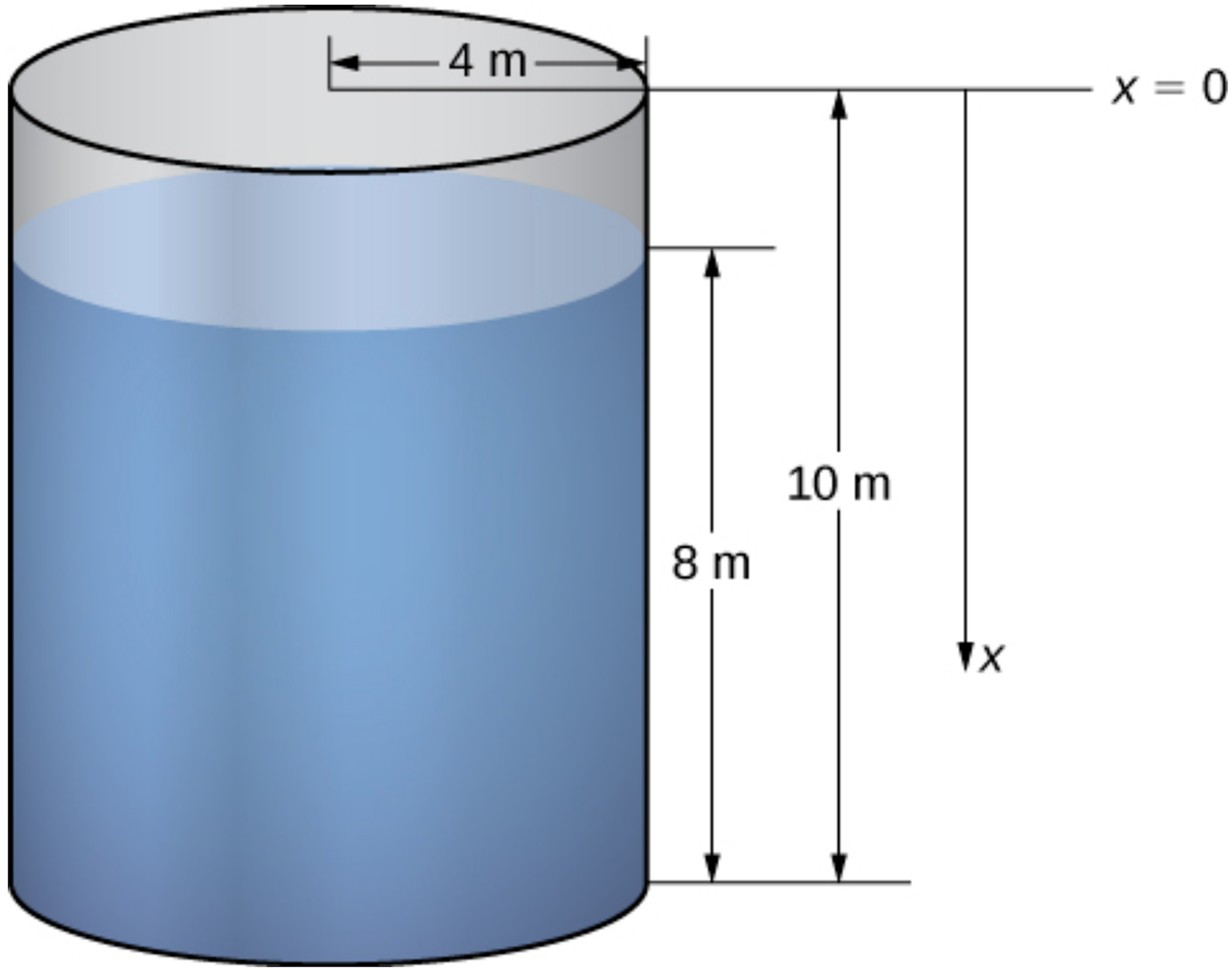


Figure 6.52 How much work is needed to empty a tank partially filled with water?

(6.5) Physical Applications

Using this coordinate system, the water extends from $x = 2$ to $x = 10$. Therefore, we partition the interval $[2,\ 10]$ and look at the work required to lift each individual "layer" of water. So, for $i = 0, 1, 2, \ldots, n,$ let $P = \{x_i\}$ be a regular partition of the interval $[2,\ 10]$, and for $i = 1, 2, \ldots, n,$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. **Figure 6.53** shows a representative layer.

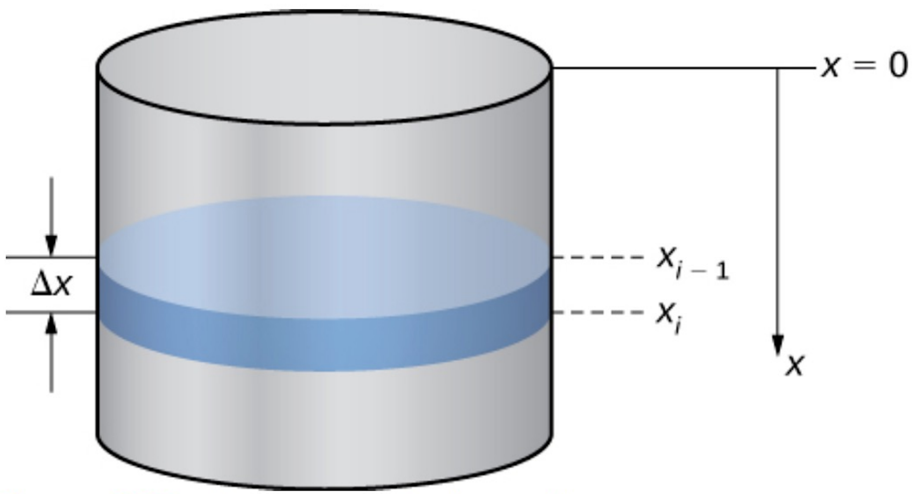


Figure 6.53 A representative layer of water.

(6.5) Physical Applications

In pumping problems, the force required to lift the water to the top of the tank is the force required to overcome gravity, so it is equal to the weight of the water. Given that the weight-density of water is $9800\ \text{N/m}^3$, or $62.4\ \text{lb/ft}^3$, calculating the volume of each layer gives us the weight. In this case, we have

$$V = \pi(4)^2\,\Delta x = 16\pi\Delta x.$$

Then, the force needed to lift each layer is

$$F = 9800 \cdot 16\pi\Delta x = 156{,}800\pi\Delta x.$$

Note that this step becomes a little more difficult if we have a noncylindrical tank. We look at a noncylindrical tank in the next example.

We also need to know the distance the water must be lifted. Based on our choice of coordinate systems, we can use x_i^* as an approximation of the distance the layer must be lifted. Then the work to lift the ith layer of water W_i is approximately

$$W_i \approx 156{,}800\pi x_i^*\ \Delta x.$$

(6.5) Physical Applications

Adding the work for each layer, we see the approximate work to empty the tank is given by

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 156{,}800\pi x_i^* \ \Delta x.$$

This is a Riemann sum, so taking the limit as $n \to \infty$, we get

$$\begin{aligned} W &= \lim_{n \to \infty} \sum_{i=1}^{n} 156{,}800\pi x_i^* \ \Delta x \\ &= 156{,}800\pi \int_2^{10} x dx \\ &= 156{,}800\pi \left[\frac{x^2}{2}\right]\Bigg|_2^{10} = 7{,}526{,}400\pi \approx 23{,}644{,}883. \end{aligned}$$

The work required to empty the tank is approximately 23,650,000 J.

(6.5) Physical Applications

Example 6.26

A Pumping Problem with a Noncylindrical Tank

Assume a tank in the shape of an inverted cone, with height 12 ft and base radius 4 ft. The tank is full to start with, and water is pumped over the upper edge of the tank until the height of the water remaining in the tank is 4 ft. How much work is required to pump out that amount of water?

Solution

The tank is depicted in **Figure 6.54**. As we did in the example with the cylindrical tank, we orient the x-axis vertically, with the origin at the top of the tank and the downward direction being positive (step 1).

(6.5) Physical Applications

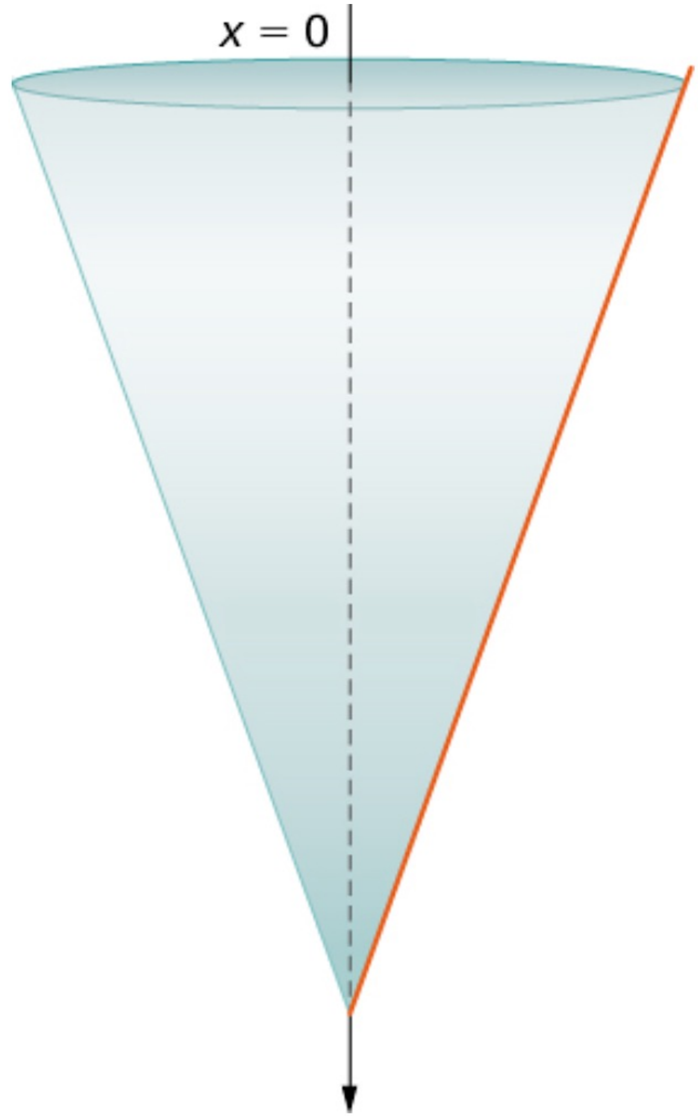


Figure 6.54 A water tank in the shape of an inverted cone.

(6.5) Physical Applications

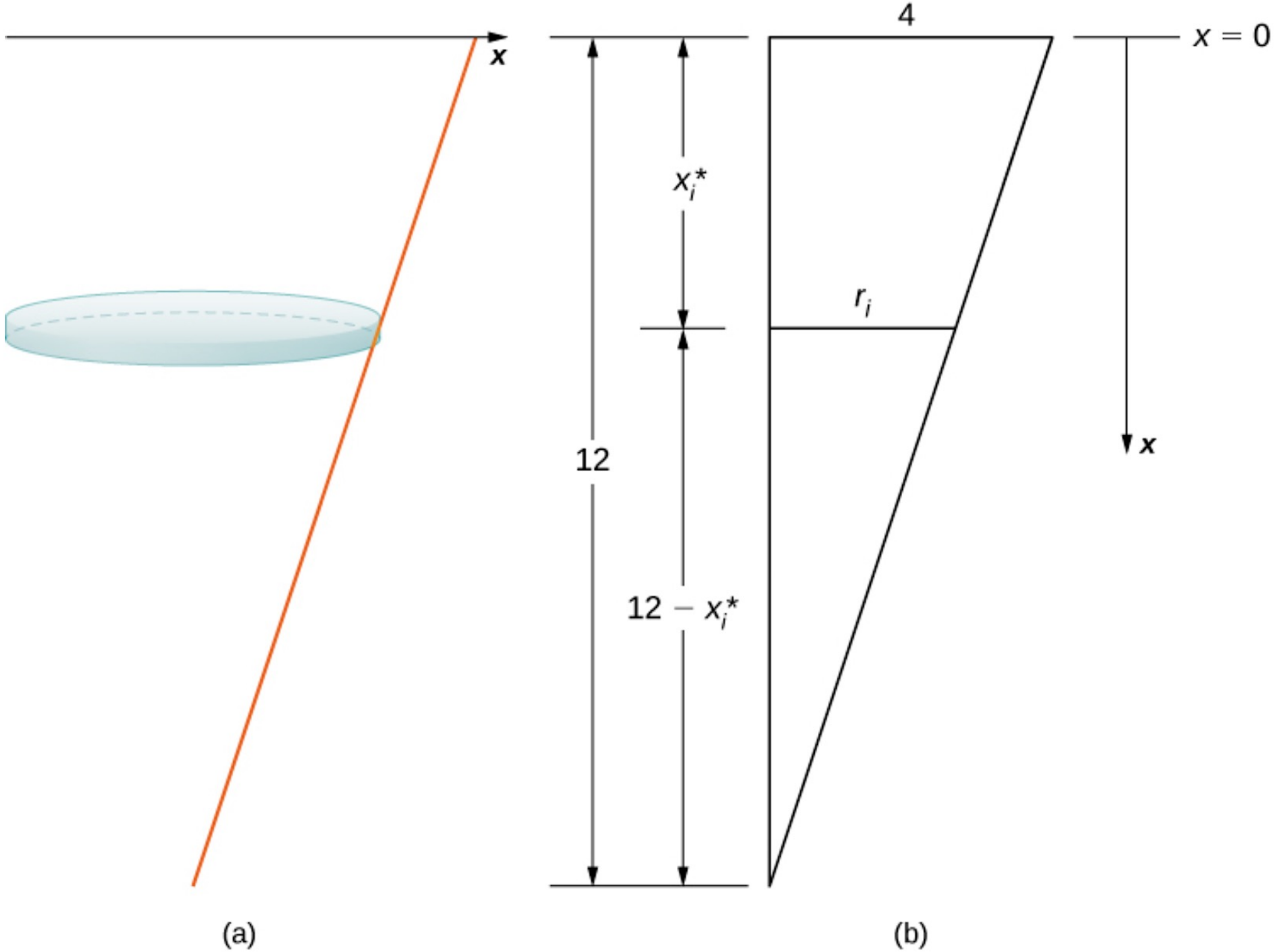


Figure 6.55 Using similar triangles to express the radius of a disk of water.

(6.5) Physical Applications

From properties of similar triangles, we have

$$\begin{aligned} \frac{r_i}{12 - x_i^*} &= \frac{4}{12} = \frac{1}{3} \\ 3r_i &= 12 - x_i^* \\ r_i &= \frac{12 - x_i^*}{3} \\ &= 4 - \frac{x_i^*}{3}. \end{aligned}$$

Then the volume of the disk is

$$V_i = \pi\left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 2).}$$

(6.5) Physical Applications

The weight-density of water is 62.4 lb/ft^3, so the force needed to lift each layer is approximately

$$F_i \approx 62.4\pi\left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 3).}$$

Based on the diagram, the distance the water must be lifted is approximately x_i^* feet (step 4), so the approximate work needed to lift the layer is

$$W_i \approx 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 5).}$$

Summing the work required to lift all the layers, we get an approximate value of the total work:

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 6).}$$

(6.5) Physical Applications

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} W &= \lim_{n \to \infty} \sum_{i=1}^{n} 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \\ &= \int_0^8 62.4\pi x \left(4 - \frac{x}{3}\right)^2 dx \\ &= 62.4\pi \int_0^8 x\left(16 - \frac{8x}{3} + \frac{x^2}{9}\right) dx = 62.4\pi \int_0^8 \left(16x - \frac{8x^2}{3} + \frac{x^3}{9}\right) dx \\ &= 62.4\pi \left[8x^2 - \frac{8x^3}{9} + \frac{x^4}{36}\right]\Bigg|_0^8 = 10{,}649.6\pi \approx 33{,}456.7. \end{aligned}$$

It takes approximately 33,450 ft-lb of work to empty the tank to the desired level.

(6.5) Physical Applications

Hydrostatic Force and Pressure

Let's begin with the simple case of a plate of area A submerged horizontally in water at a depth s (Figure 6.56). Then, the force exerted on the plate is simply the weight of the water above it, which is given by $F = \rho As,$ where ρ is the weight density of water (weight per unit volume). To find the **hydrostatic pressure**—that is, the pressure exerted by water on a submerged object—we divide the force by the area. So the pressure is $p = F/A = \rho s.$

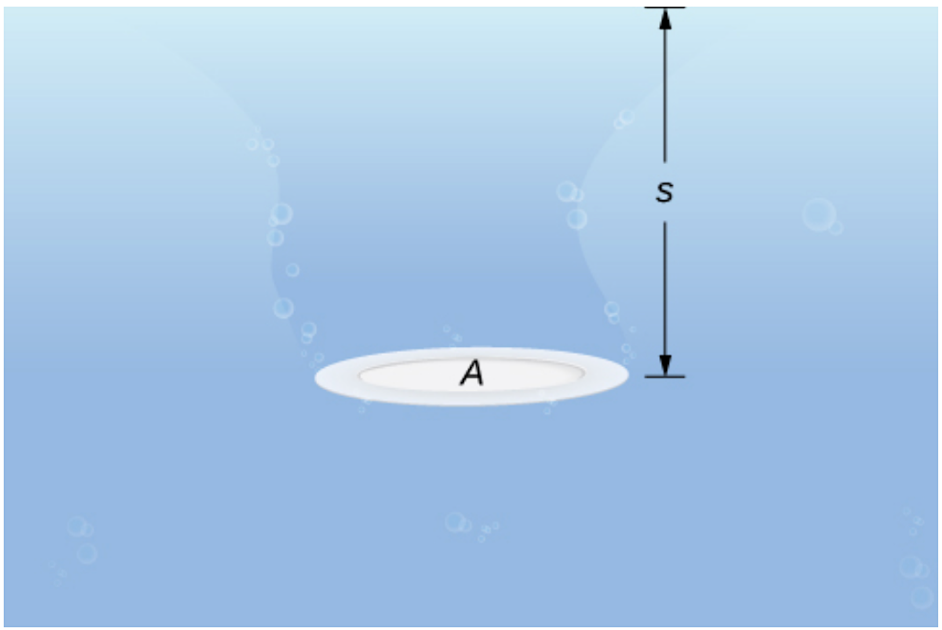


Figure 6.56 A plate submerged horizontally in water.

(6.5) Physical Applications

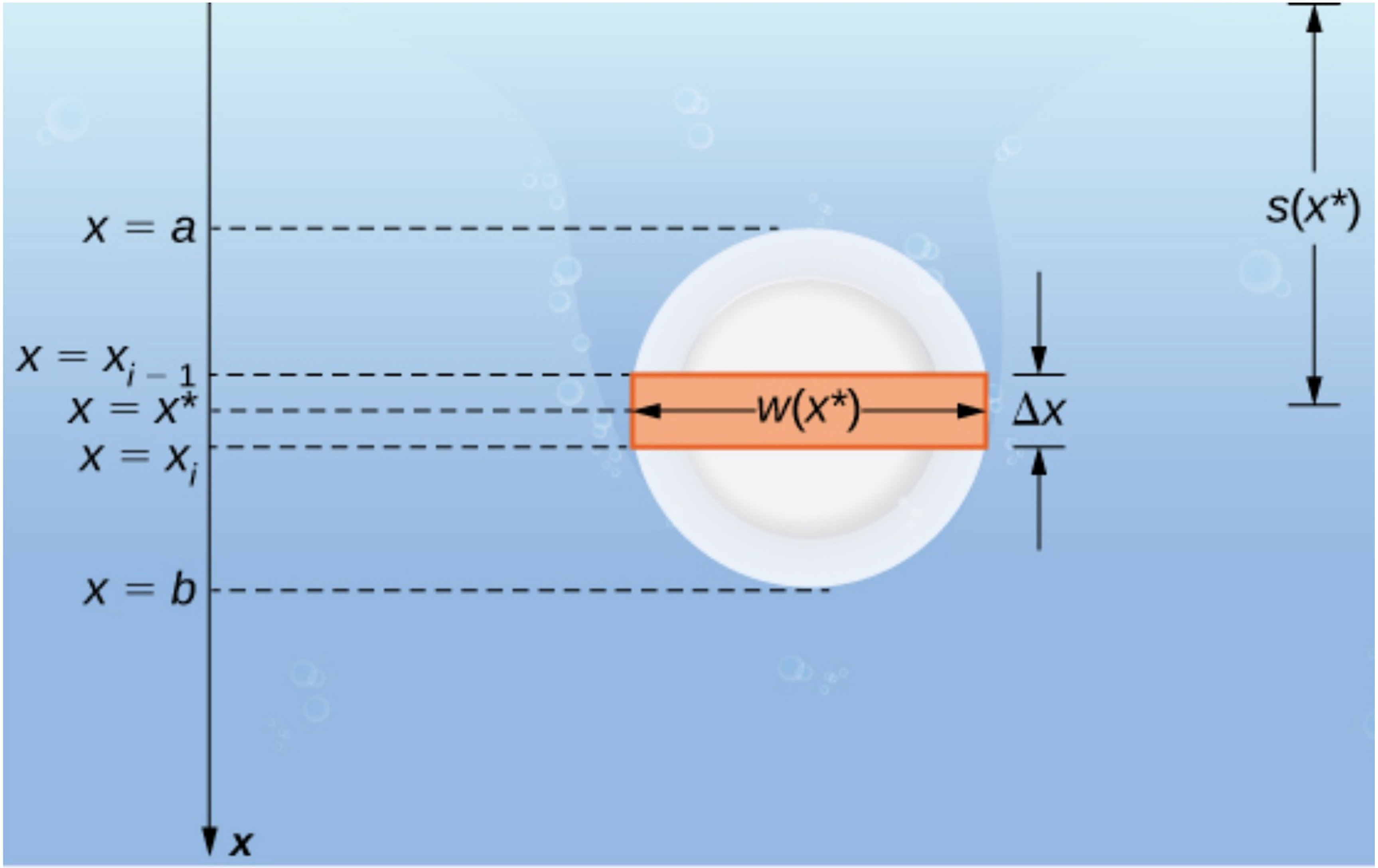


Figure 6.57 A thin plate submerged vertically in water.

(6.5) Physical Applications

Let’s now estimate the force on a representative strip. If the strip is thin enough, we can treat it as if it is at a constant depth, $s(x_i^*)$. We then have

$$F_i = \rho A s = \rho\left[w(x_i^*)\Delta x\right]s(x_i^*).$$

Adding the forces, we get an estimate for the force on the plate:

$$F \approx \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \rho\left[w(x_i^*)\Delta x\right]s(x_i^*).$$

This is a Riemann sum, so taking the limit gives us the exact force. We obtain

$$F = \lim_{n \to \infty} \sum_{i=1}^{n} \rho\left[w(x_i^*)\Delta x\right]s(x_i^*) = \int_a^b \rho w(x)s(x)dx. \qquad \textbf{(6.13)}$$

(6.5) Physical Applications

Example 6.27

Finding Hydrostatic Force

A water trough 15 ft long has ends shaped like inverted isosceles triangles, with base 8 ft and height 3 ft. Find the force on one end of the trough if the trough is full of water.

Solution

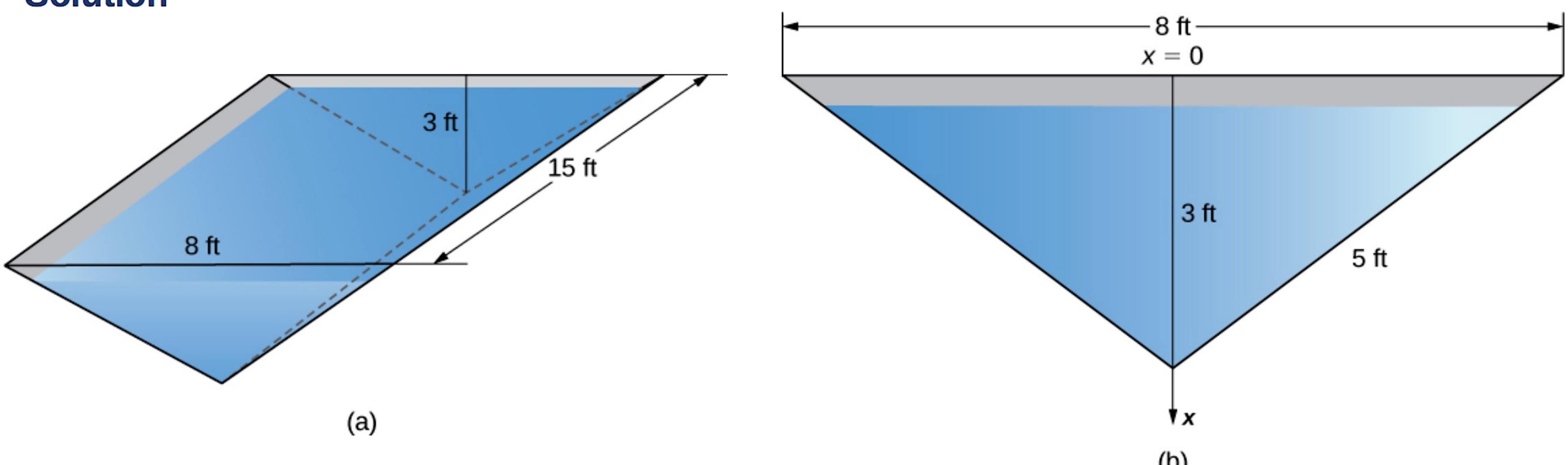


Figure 6.58 (a) A water trough with a triangular cross-section. (b) Dimensions of one end of the water trough.

(6.5) Physical Applications

Select a frame of reference with the x-axis oriented vertically and the downward direction being positive. Select the top of the trough as the point corresponding to $x = 0$ (step 1). The depth function, then, is $s(x) = x$. Using similar triangles, we see that $w(x) = 8 - (8/3)x$ (step 2). Now, the weight density of water is $62.4\ \text{lb/ft}^3$ (step 3), so applying **Equation 6.13**, we obtain

$$\begin{aligned} F &= \int_a^b \rho w(x)s(x)dx \\ &= \int_0^3 62.4\left(8 - \frac{8}{3}x\right)x\,dx = 62.4\int_0^3 \left(8x - \frac{8}{3}x^2\right)dx \\ &= 62.4\left[4x^2 - \frac{8}{9}x^3\right]\Big|_0^3 = 748.8. \end{aligned}$$

The water exerts a force of 748.8 lb on the end of the trough (step 4).

(6.6) Moments and Centers of Mass

Center of Mass and Moments

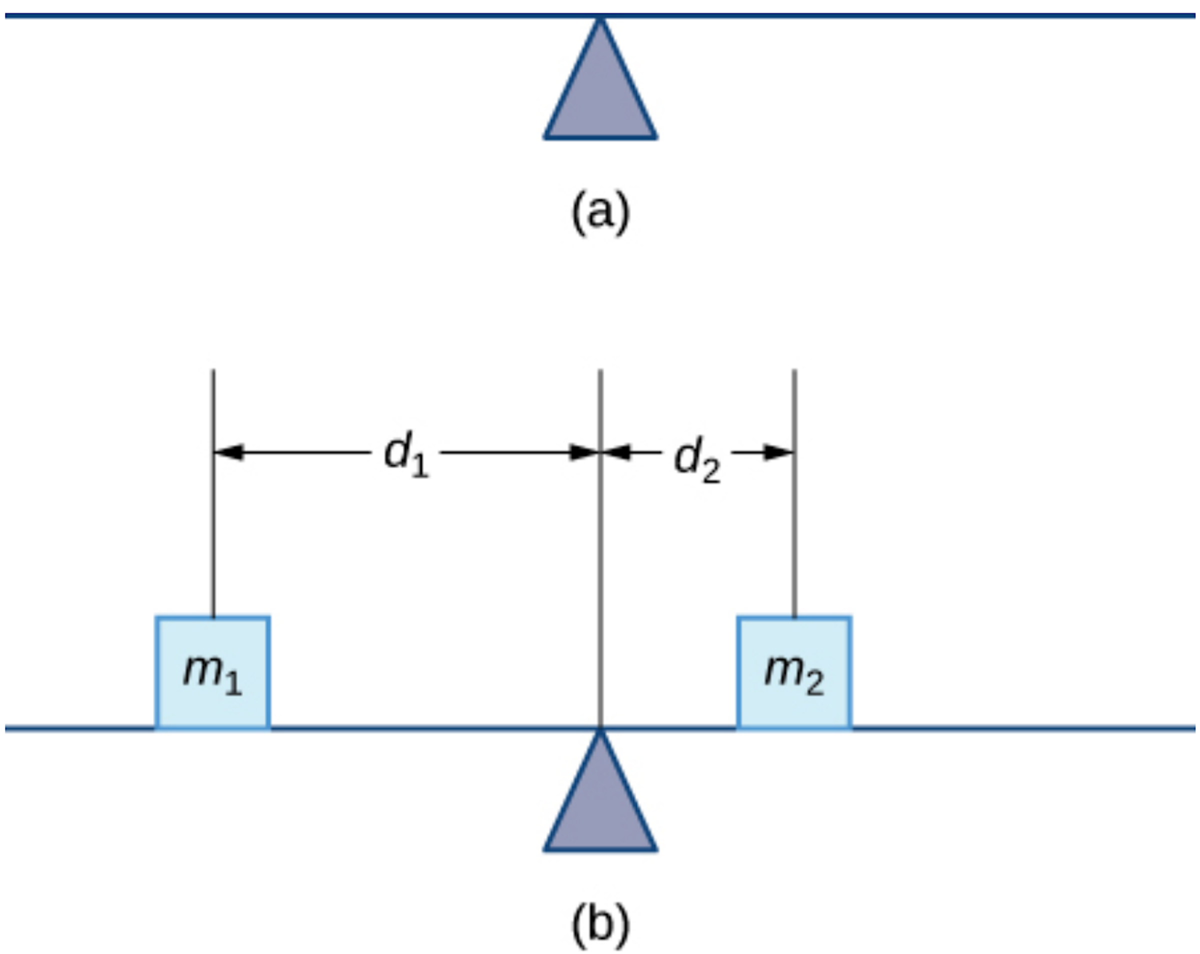


Figure 6.62 (a) A thin rod rests on a fulcrum. (b) Masses are placed on the rod.

(6.6) Moments and Centers of Mass

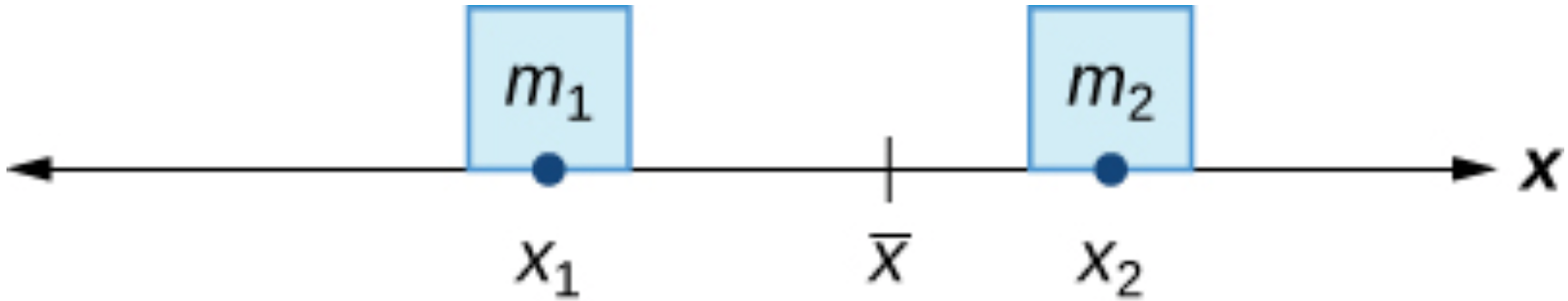


Figure 6.63 The center of mass $\bar{x}$ is the balance point of the system.

(6.6) Moments and Centers of Mass

$$
\begin{aligned}
m_1 |x_1 - \bar{x}| &= m_2 |x_2 - \bar{x}| \\
m_1 (\bar{x} - x_1) &= m_2 (x_2 - \bar{x}) \\
m_1 \bar{x} - m_1 x_1 &= m_2 x_2 - m_2 \bar{x} \\
\bar{x} (m_1 + m_2) &= m_1 x_1 + m_2 x_2 \\
\bar{x} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.
\end{aligned}
$$

(6.6) Moments and Centers of Mass

The expression in the numerator, $m_1 x_1 + m_2 x_2,$ is called the *first moment of the system with respect to the origin*. If the context is clear, we often drop the word *first* and just refer to this expression as the **moment** of the system. The expression in the denominator, $m_1 + m_2,$ is the total mass of the system. Thus, the **center of mass** of the system is the point at which the total mass of the system could be concentrated without changing the moment.

This idea is not limited just to two point masses. In general, if *n* masses, $m_1, m_2, \ldots, m_n,$ are placed on a number line at points $x_1, x_2, \ldots, x_n,$ respectively, then the center of mass of the system is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}.$$

(6.6) Moments and Centers of Mass

Theorem 6.9: Center of Mass of Objects on a Line

Let $m_1, m_2, \ldots, m_n$ be point masses placed on a number line at points $x_1, x_2, \ldots, x_n,$ respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then, the moment of the system with respect to the origin is given by

$$M = \sum_{i=1}^{n} m_i x_i \tag{6.14}$$

and the center of mass of the system is given by

$$\bar{x} = \frac{M}{m}. \tag{6.15}$$

(6.6) Moments and Centers of Mass

Example 6.29

Finding the Center of Mass of Objects along a Line

Suppose four point masses are placed on a number line as follows:

$$m_1 = 30\,\text{kg, placed at } x_1 = -2\,\text{m} \qquad m_2 = 5\,\text{kg, placed at } x_2 = 3\,\text{m}$$
$$m_3 = 10\,\text{kg, placed at } x_3 = 6\,\text{m} \qquad m_4 = 15\,\text{kg, placed at } x_4 = -3\,\text{m.}$$

Find the moment of the system with respect to the origin and find the center of mass of the system.

(6.6) Moments and Centers of Mass

Solution

First, we need to calculate the moment of the system:

$$\begin{aligned} M &= \sum_{i=1}^{4} m_i x_i \\ &= -60 + 15 + 60 - 45 = -30. \end{aligned}$$

Now, to find the center of mass, we need the total mass of the system:

$$\begin{aligned} m &= \sum_{i=1}^{4} m_i \\ &= 30 + 5 + 10 + 15 = 60 \text{ kg}. \end{aligned}$$

Then we have

$$\bar{x} = \frac{M}{m} = \frac{-30}{60} = -\frac{1}{2}.$$

The center of mass is located 1/2 m to the left of the origin.

(6.6) Moments and Centers of Mass

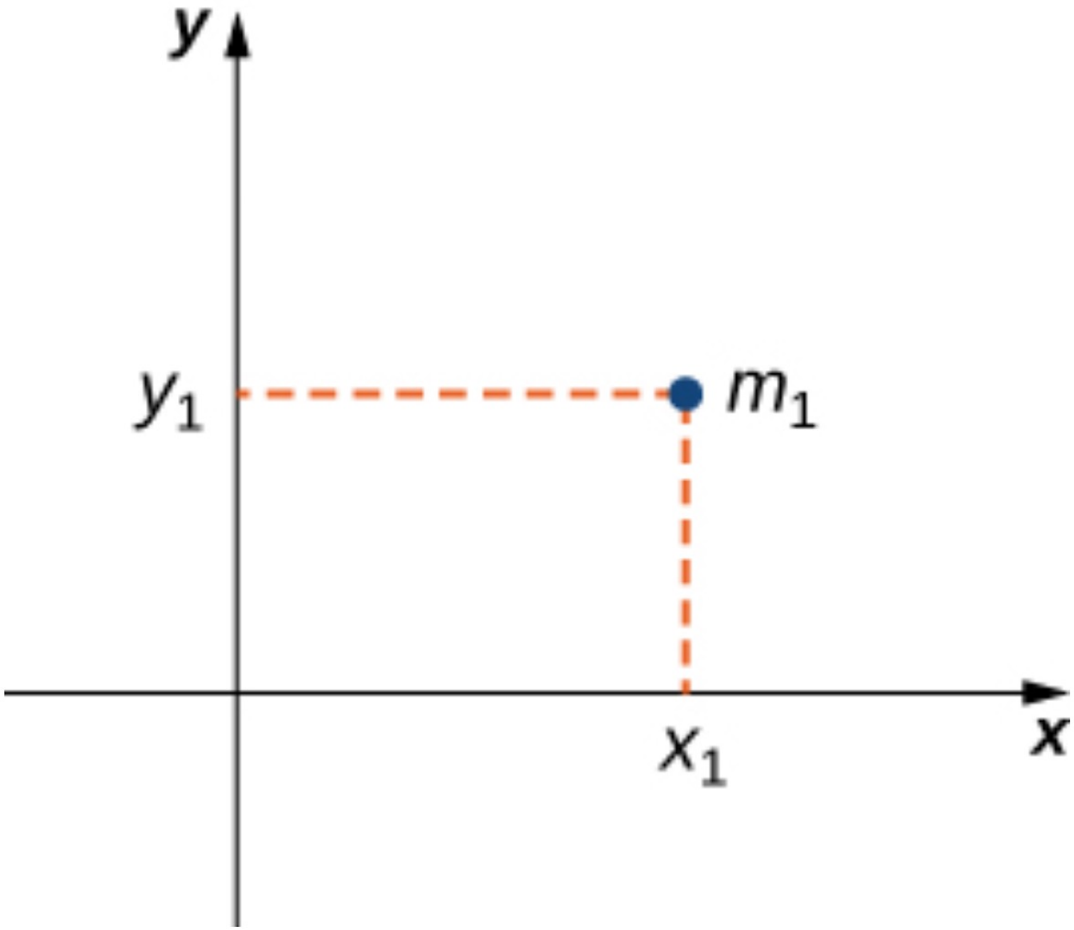


Figure 6.64 Point mass m_1 is located at point (x_1, y_1) in the plane.

(6.6) Moments and Centers of Mass

Theorem 6.10: Center of Mass of Objects in a Plane

Let $m_1, m_2, \ldots, m_n$ be point masses located in the *xy*-plane at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then the moments M_x and M_y of the system with respect to the *x*- and *y*-axes, respectively, are given by

$$M_x = \sum_{i=1}^{n} m_i y_i \quad \text{and} \quad M_y = \sum_{i=1}^{n} m_i x_i. \tag{6.16}$$

Also, the coordinates of the center of mass $(\bar{x}, \bar{y})$ of the system are

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}. \tag{6.17}$$

(6.6) Moments and Centers of Mass

Example 6.30

Finding the Center of Mass of Objects in a Plane

Suppose three point masses are placed in the *xy*-plane as follows (assume coordinates are given in meters):

$$m_1 = 2\,\text{kg}, \text{ placed at } (-1,\ 3),$$
$$m_2 = 6\,\text{kg}, \text{ placed at } (1,\ 1),$$
$$m_3 = 4\,\text{kg}, \text{ placed at } (2,\ -2).$$

Find the center of mass of the system.

(6.6) Moments and Centers of Mass

Solution

First we calculate the total mass of the system:

$$m = \sum_{i=1}^{3} m_i = 2 + 6 + 4 = 12\,\text{kg}.$$

Next we find the moments with respect to the x- and y-axes:

$$M_y = \sum_{i=1}^{3} m_i x_i = -2 + 6 + 8 = 12,$$

$$M_x = \sum_{i=1}^{3} m_i y_i = 6 + 6 - 8 = 4.$$

Then we have

$$\bar{x} = \frac{M_y}{m} = \frac{12}{12} = 1 \text{ and } \bar{y} = \frac{M_x}{m} = \frac{4}{12} = \frac{1}{3}.$$

The center of mass of the system is $(1, 1/3)$, in meters.

(6.6) Moments and Centers of Mass

Center of Mass of Thin Plates

Theorem 6.11: The Symmetry Principle

If a region R is symmetric about a line l, then the centroid of R lies on l.

(6.6) Moments and Centers of Mass

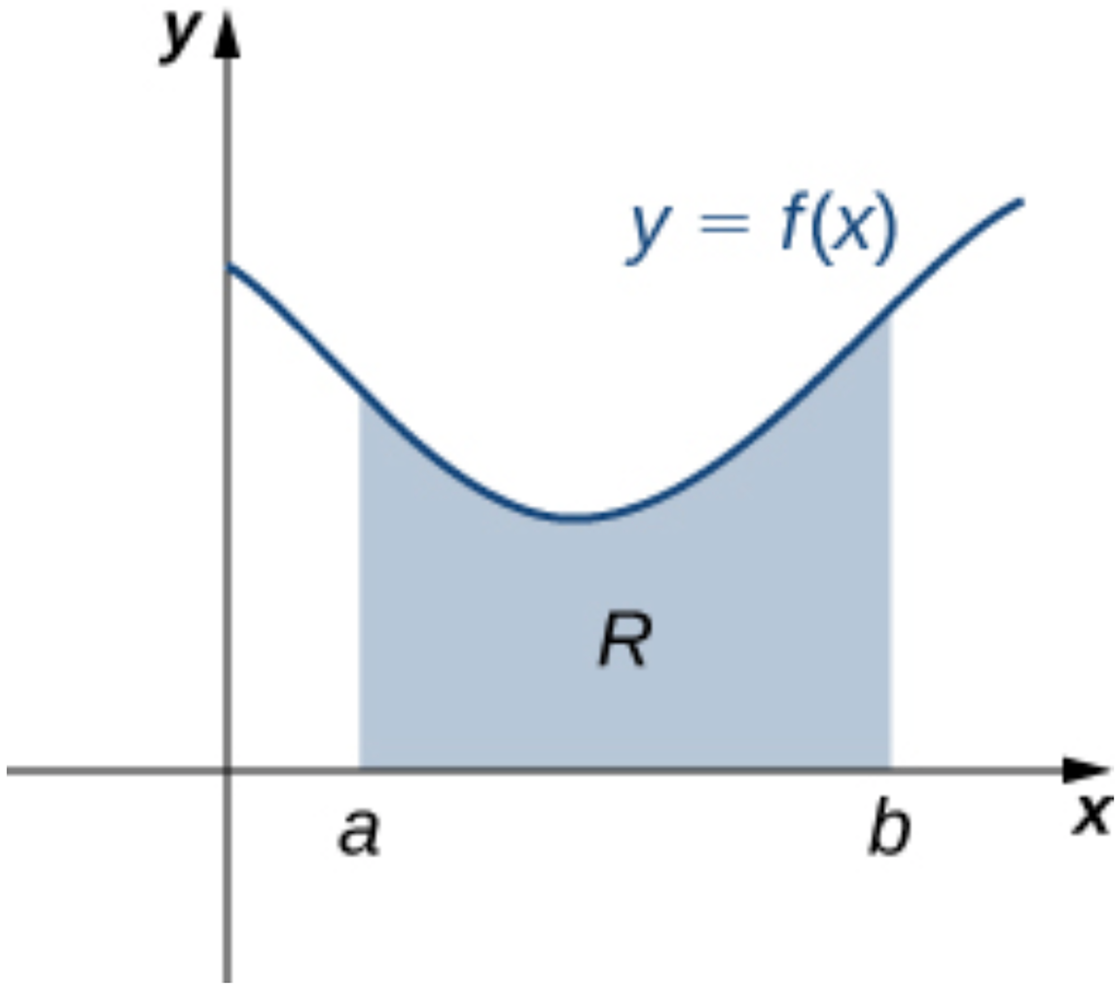


Figure 6.65 A region in the plane representing a lamina.

(6.6) Moments and Centers of Mass

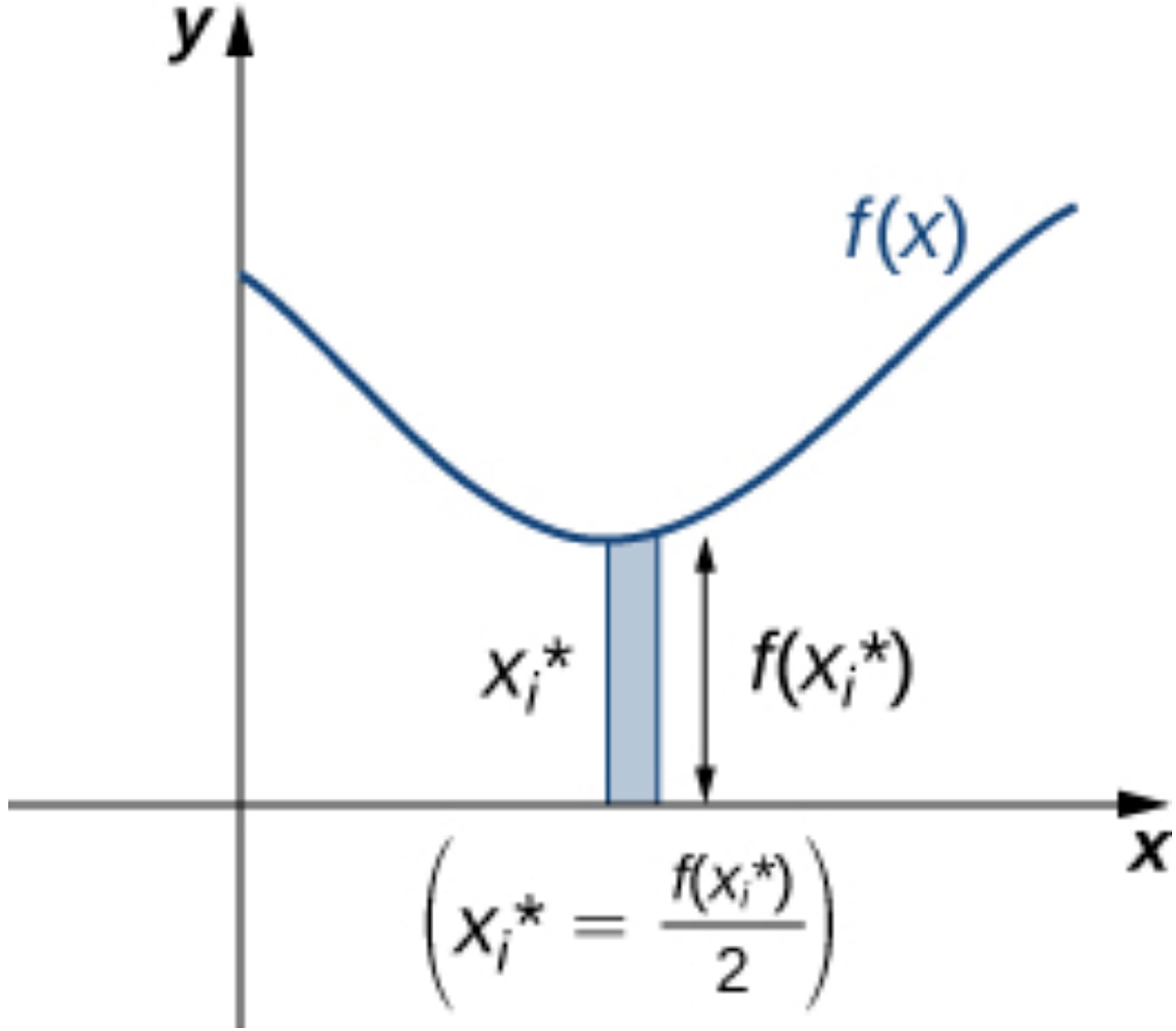


Figure 6.66 A representative rectangle of the lamina.

(6.6) Moments and Centers of Mass

Next, we need to find the total mass of the rectangle. Let ρ represent the density of the lamina (note that ρ is a constant). In this case, ρ is expressed in terms of mass per unit area. Thus, to find the total mass of the rectangle, we multiply the area of the rectangle by ρ. Then, the mass of the rectangle is given by $\rho f(x_i^*)\Delta x$.

To get the approximate mass of the lamina, we add the masses of all the rectangles to get

$$m \approx \sum_{i=1}^{n} \rho f(x_i^*)\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$ gives the exact mass of the lamina:

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho f(x_i^*)\Delta x = \rho \int_a^b f(x)dx.$$

(6.6) Moments and Centers of Mass

Next, we calculate the moment of the lamina with respect to the x-axis. Returning to the representative rectangle, recall its center of mass is $\left(x_i^*, \left(f(x_i^*)\right)/2\right)$. Recall also that treating the rectangle as if it is a point mass located at the center of mass does not change the moment. Thus, the moment of the rectangle with respect to the x-axis is given by the mass of the rectangle, $\rho f(x_i^*)\Delta x$, multiplied by the distance from the center of mass to the x-axis: $\left(f(x_i^*)\right)/2$. Therefore, the moment with respect to the x-axis of the rectangle is $\rho\left(\left[f(x_i^*)\right]^2/2\right)\Delta x$. Adding the moments of the rectangles and taking the limit of the resulting Riemann sum, we see that the moment of the lamina with respect to the x-axis is

$$M_x = \lim_{n \to \infty} \sum_{i=1}^{n} \rho \frac{\left[f(x_i^*)\right]^2}{2} \Delta x = \rho \int_a^b \frac{\left[f(x)\right]^2}{2} dx.$$

(6.6) Moments and Centers of Mass

We derive the moment with respect to the y-axis similarly, noting that the distance from the center of mass of the rectangle to the y-axis is x_i^*. Then the moment of the lamina with respect to the y-axis is given by

$$M_y = \lim_{n \to \infty} \sum_{i=1}^{n} \rho x_i^* f(x_i^*) \Delta x = \rho \int_a^b x f(x) dx.$$

We find the coordinates of the center of mass by dividing the moments by the total mass to give $\bar{x} = M_y/m$ and $\bar{y} = M_x/m$. If we look closely at the expressions for M_x, M_y, and m, we notice that the constant ρ cancels out when $\bar{x}$ and $\bar{y}$ are calculated.

(6.6) Moments and Centers of Mass

Theorem 6.12: Center of Mass of a Thin Plate in the *xy*-Plane

Let R denote a region bounded above by the graph of a continuous function $f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho \int_a^b f(x)dx. \tag{6.18}$$

ii. The moments M_x and M_y of the lamina with respect to the x- and y-axes, respectively, are

$$M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx \text{ and } M_y = \rho \int_a^b x f(x)dx. \tag{6.19}$$

iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}. \tag{6.20}$$

(6.6) Moments and Centers of Mass

Example 6.31

Finding the Center of Mass of a Lamina

Let R be the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the x-axis over the interval $[0, 4]$. Find the centroid of the region.

Solution

The region is depicted in the following figure.

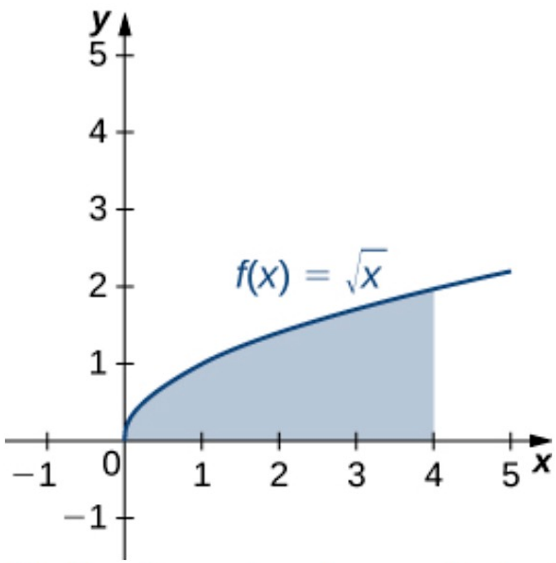


Figure 6.67 Finding the center of mass of a lamina.

(6.6) Moments and Centers of Mass

Since we are only asked for the centroid of the region, rather than the mass or moments of the associated lamina, we know the density constant ρ cancels out of the calculations eventually. Therefore, for the sake of convenience, let's assume $\rho = 1$.

First, we need to calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b f(x)dx = \int_0^4 \sqrt{x}\,dx \\ &= \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{2}{3}[8 - 0] = \frac{16}{3}. \end{aligned}$$

Next, we compute the moments:

$$\begin{aligned} M_x &= \rho\int_a^b \frac{[f(x)]^2}{2}dx \\ &= \int_0^4 \frac{x}{2}dx = \frac{1}{4}x^2\Big|_0^4 = 4 \end{aligned}$$

and

$$\begin{aligned} M_y &= \rho\int_a^b xf(x)dx \\ &= \int_0^4 x\sqrt{x}dx = \int_0^4 x^{3/2}dx \\ &= \frac{2}{5}x^{5/2}\Big|_0^4 = \frac{2}{5}[32-0] = \frac{64}{5}. \end{aligned}$$

Thus, we have

$$\bar{x} = \frac{M_y}{m} = \frac{64/5}{16/3} = \frac{64}{5}\cdot\frac{3}{16} = \frac{12}{5} \text{ and } \bar{y} = \frac{M_x}{y} = \frac{4}{16/3} = 4\cdot\frac{3}{16} = \frac{3}{4}.$$

The centroid of the region is $(12/5,\ 3/4)$.

(6.6) Moments and Centers of Mass

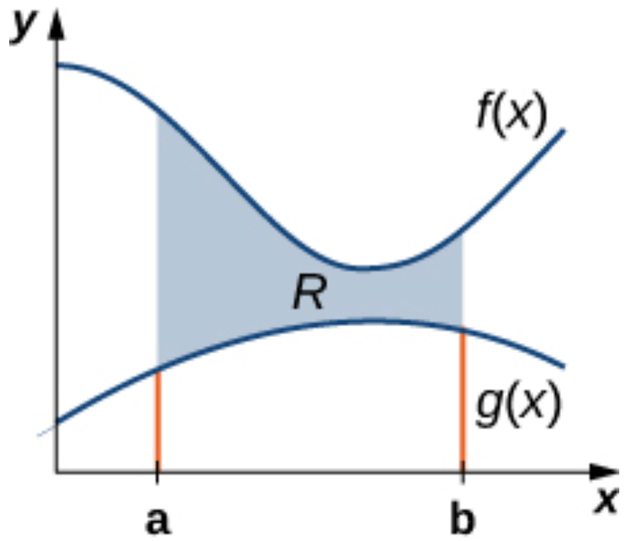


Figure 6.68 A region between two functions.

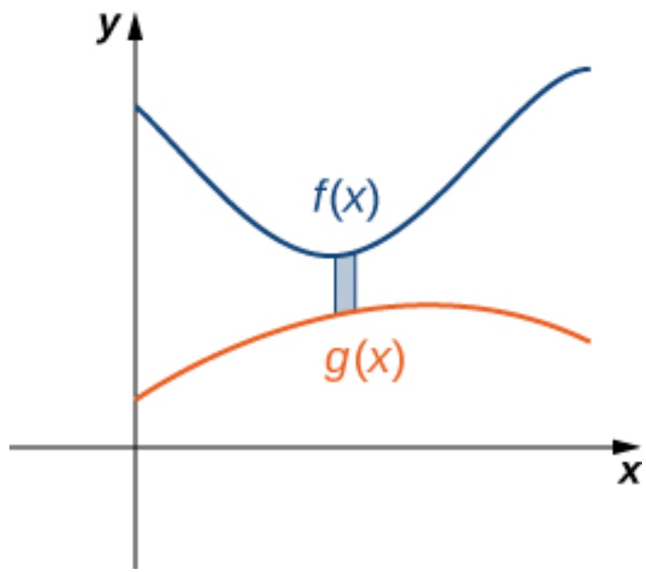


Figure 6.69 A representative rectangle of the region between two functions.

(6.6) Moments and Centers of Mass

Theorem 6.13: Center of Mass of a Lamina Bounded by Two Functions

Let R denote a region bounded above by the graph of a continuous function $f(x)$, below by the graph of the continuous function $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho \int_a^b [f(x) - g(x)]dx. \tag{6.21}$$

ii. The moments M_x and M_y of the lamina with respect to the x- and y-axes, respectively, are

$$M_x = \rho \int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right)dx \text{ and } M_y = \rho \int_a^b x[f(x) - g(x)]dx. \tag{6.22}$$

iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}. \tag{6.23}$$

(6.6) Moments and Centers of Mass

Example 6.32

Finding the Centroid of a Region Bounded by Two Functions

Let R be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by the graph of the function $g(x) = x - 1$. Find the centroid of the region.

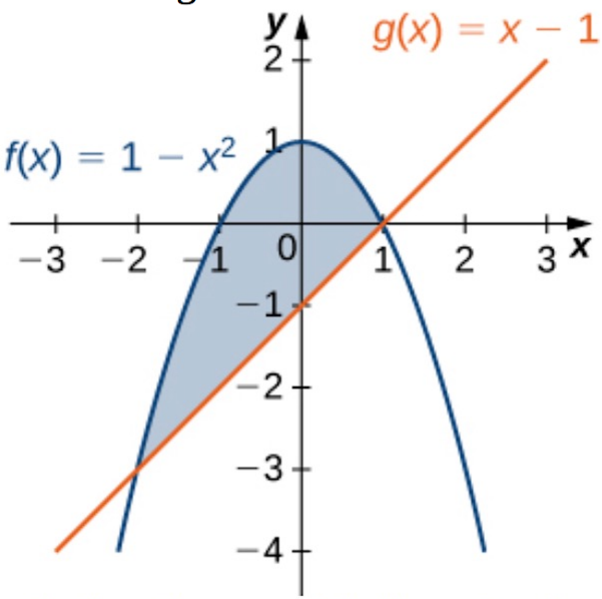


Figure 6.70 Finding the centroid of a region between two curves.

The graphs of the functions intersect at $(-2, -3)$ and $(1, 0)$, so we integrate from −2 to 1. Once again, for the sake of convenience, assume $\rho = 1$.

(6.6) Moments and Centers of Mass

First, we need to calculate the total mass:

$$\begin{aligned} m &= \rho\int_a^b [f(x) - g(x)]dx \\ &= \int_{-2}^{1}\left[1 - x^2 - (x-1)\right]dx = \int_{-2}^{1}(2 - x^2 - x)dx \\ &= \left[2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right]\Big|_{-2}^{1} = \left[2 - \frac{1}{3} - \frac{1}{2}\right] - \left[-4 + \frac{8}{3} - 2\right] = \frac{9}{2}. \end{aligned}$$

(6.6) Moments and Centers of Mass

Next, we compute the moments:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right)dx \\ &= \frac{1}{2}\int_{-2}^{1}\left(\left(1 - x^2\right)^2 - (x-1)^2\right)dx = \frac{1}{2}\int_{-2}^{1}\left(x^4 - 3x^2 + 2x\right)dx \\ &= \frac{1}{2}\left[\frac{x^5}{5} - x^3 + x^2\right]\Bigg|_{-2}^{1} = -\frac{27}{10} \end{aligned}$$

(6.6) Moments and Centers of Mass

and

$$\begin{aligned} M_y &= \rho\int_a^b x[f(x) - g(x)]dx \\ &= \int_{-2}^{1} x\left[(1 - x^2) - (x - 1)\right]dx = \int_{-2}^{1} x\left[2 - x^2 - x\right]dx = \int_{-2}^{1}\left(2x - x^4 - x^2\right)dx \\ &= \left[x^2 - \frac{x^5}{5} - \frac{x^3}{3}\right]\Bigg|_{-2}^{1} = -\frac{9}{4}. \end{aligned}$$

Therefore, we have

$$\bar{x} = \frac{M_y}{m} = -\frac{9}{4}\cdot\frac{2}{9} = -\frac{1}{2} \text{ and } \bar{y} = \frac{M_x}{y} = -\frac{27}{10}\cdot\frac{2}{9} = -\frac{3}{5}.$$

The centroid of the region is $(-(1/2), -(3/5))$.

(6.6) Moments and Centers of Mass

The Symmetry Principle

Example 6.33

Finding the Centroid of a Symmetric Region

Let R be the region bounded above by the graph of the function $f(x) = 4 - x^2$ and below by the x-axis. Find the centroid of the region.

Solution

The region is depicted in the following figure.

(6.6) Moments and Centers of Mass

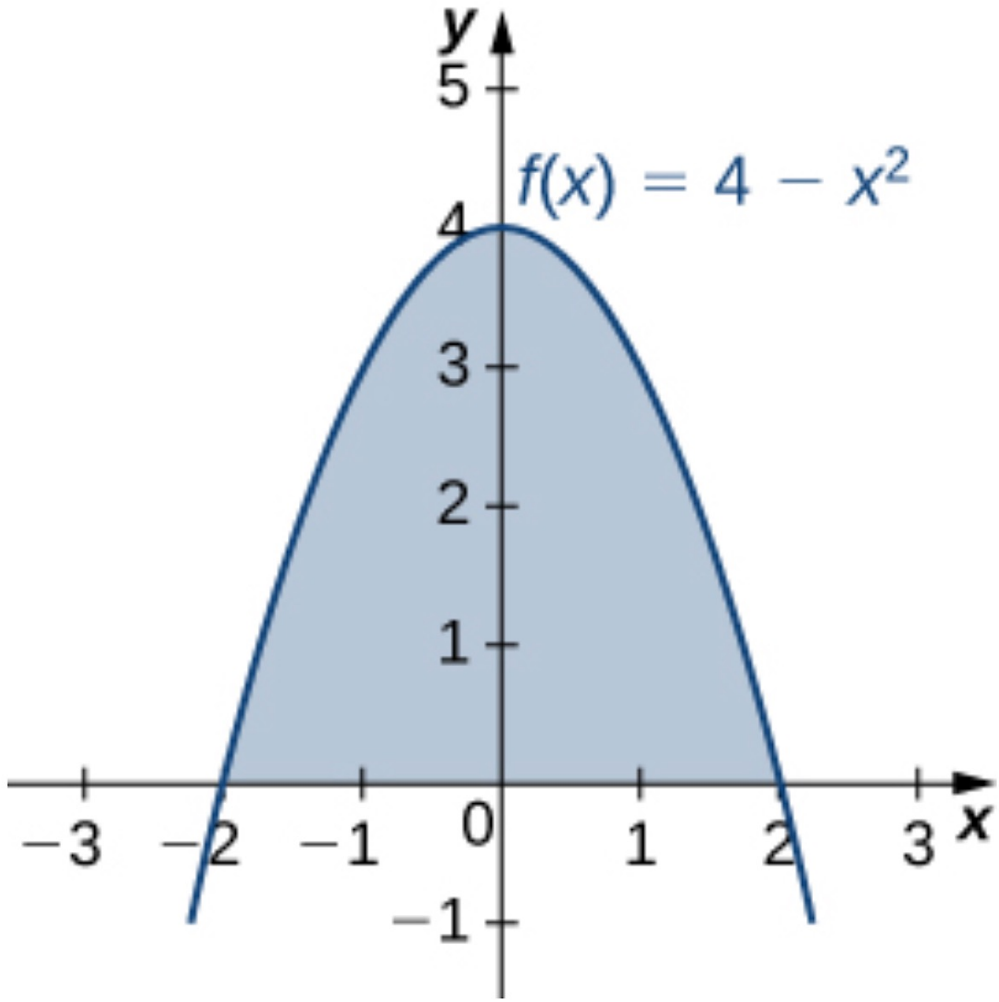


Figure 6.71 We can use the symmetry principle to help find the centroid of a symmetric region.

(6.6) Moments and Centers of Mass

The region is symmetric with respect to the y-axis. Therefore, the x-coordinate of the centroid is zero. We need only calculate $\bar{y}$. Once again, for the sake of convenience, assume $\rho = 1$.

First, we calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b f(x)dx \\ &= \int_{-2}^{2} \left(4 - x^2\right) dx \\ &= \left[4x - \frac{x^3}{3}\right]\Bigg|_{-2}^{2} = \frac{32}{3}. \end{aligned}$$

(6.6) Moments and Centers of Mass

Next, we calculate the moments. We only need M_x:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{[f(x)]^2}{2} dx \\ &= \frac{1}{2} \int_{-2}^{2} \left[4 - x^2\right]^2 dx = \frac{1}{2} \int_{-2}^{2} \left(16 - 8x^2 + x^4\right) dx \\ &= \frac{1}{2} \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]\Bigg|_{-2}^{2} = \frac{256}{15}. \end{aligned}$$

Then we have

$$\bar{y} = \frac{M_x}{y} = \frac{256}{15} \cdot \frac{3}{32} = \frac{8}{5}.$$

The centroid of the region is $(0,\ 8/5)$.

(6.6) Moments and Centers of Mass

Theorem of Pappus

This section ends with a discussion of the **theorem of Pappus for volume**, which allows us to find the volume of particular kinds of solids by using the centroid. (There is also a theorem of Pappus for surface area, but it is much less useful than the theorem for volume.)

Theorem 6.14: Theorem of Pappus for Volume

Let R be a region in the plane and let l be a line in the plane that does not intersect R. Then the volume of the solid of revolution formed by revolving R around l is equal to the area of R multiplied by the distance d traveled by the centroid of R.

(6.6) Moments and Centers of Mass

Proof

We can prove the case when the region is bounded above by the graph of a function $f(x)$ and below by the graph of a function $g(x)$ over an interval $[a, b]$, and for which the axis of revolution is the y-axis. In this case, the area of the region is $A = \int_a^b [f(x) - g(x)]dx$. Since the axis of rotation is the y-axis, the distance traveled by the centroid of the region depends only on the x-coordinate of the centroid, $\bar{x}$, which is

$$\bar{x} = \frac{M_y}{m},$$

where

$$m = \rho \int_a^b [f(x) - g(x)]dx \text{ and } M_y = \rho \int_a^b x[f(x) - g(x)]dx.$$

(6.6) Moments and Centers of Mass

Then,

$$d = 2\pi \frac{\rho \int_a^b x[f(x) - g(x)]dx}{\rho \int_a^b [f(x) - g(x)]dx}$$

and thus

$$d \cdot A = 2\pi \int_a^b x[f(x) - g(x)]dx.$$

However, using the method of cylindrical shells, we have

$$V = 2\pi \int_a^b x[f(x) - g(x)]dx.$$

So,

$$V = d \cdot A$$

and the proof is complete.

(6.6) Moments and Centers of Mass

Example 6.34

Using the Theorem of Pappus for Volume

Let R be a circle of radius 2 centered at $(4, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving R around the y-axis.

(6.6) Moments and Centers of Mass

Solution

The region and torus are depicted in the following figure.

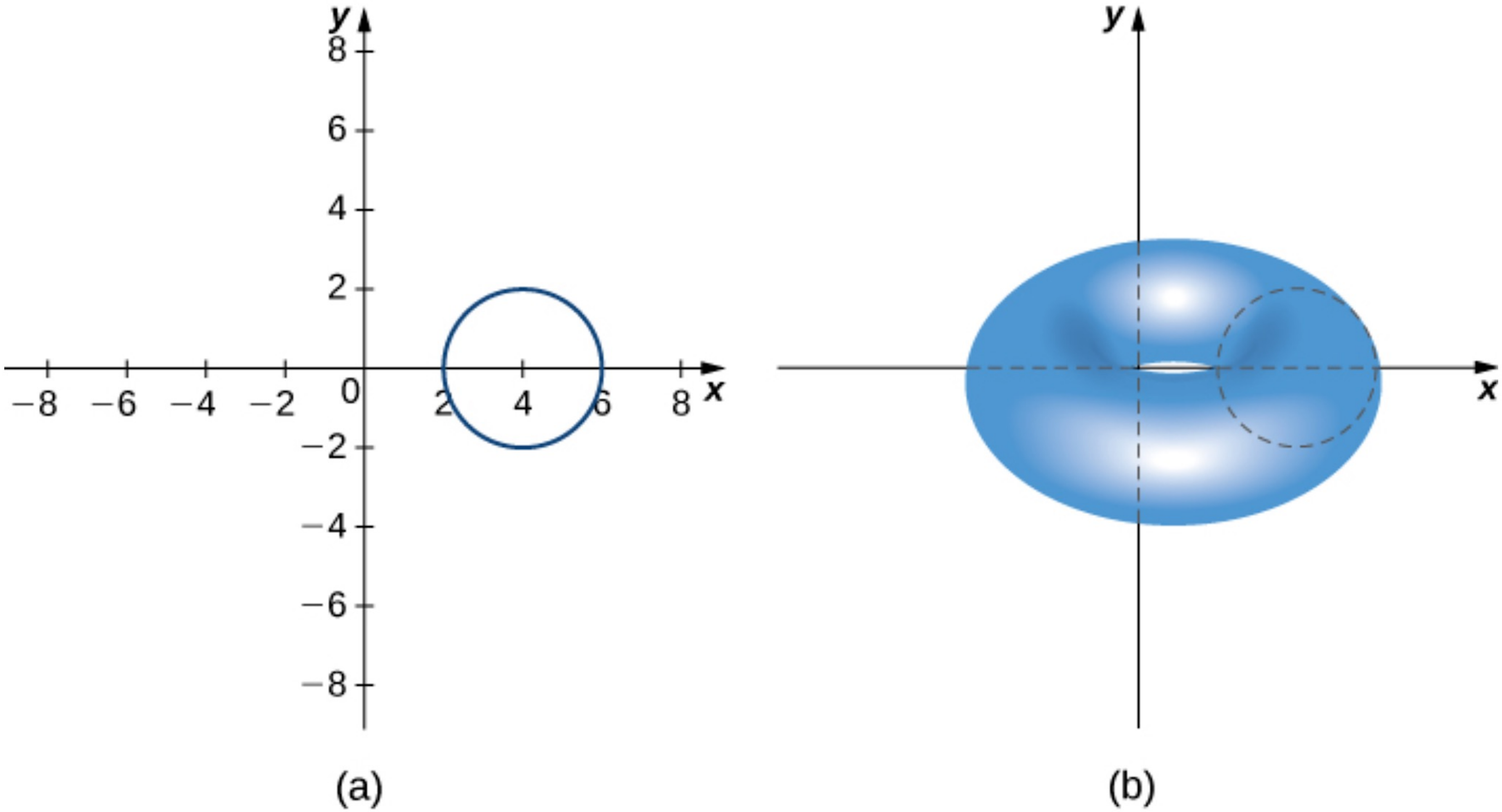


Figure 6.74 Determining the volume of a torus by using the theorem of Pappus. (a) A circular region R in the plane; (b) the torus generated by revolving R about the y-axis.

(6.6) Moments and Centers of Mass

The region R is a circle of radius 2, so the area of R is $A = 4\pi$ units^2. By the symmetry principle, the centroid of R is the center of the circle. The centroid travels around the y-axis in a circular path of radius 4, so the centroid travels $d = 8\pi$ units. Then, the volume of the torus is $A \cdot d = 32\pi^2$ units^3.

(6.8) Exponential Growth and Decay

Exponential Growth Model

Rule: Exponential Growth Model

Systems that exhibit **exponential growth** increase according to the mathematical model

$$y = y_0 e^{kt},$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the *growth constant*.

(6.8) Exponential Growth and Decay

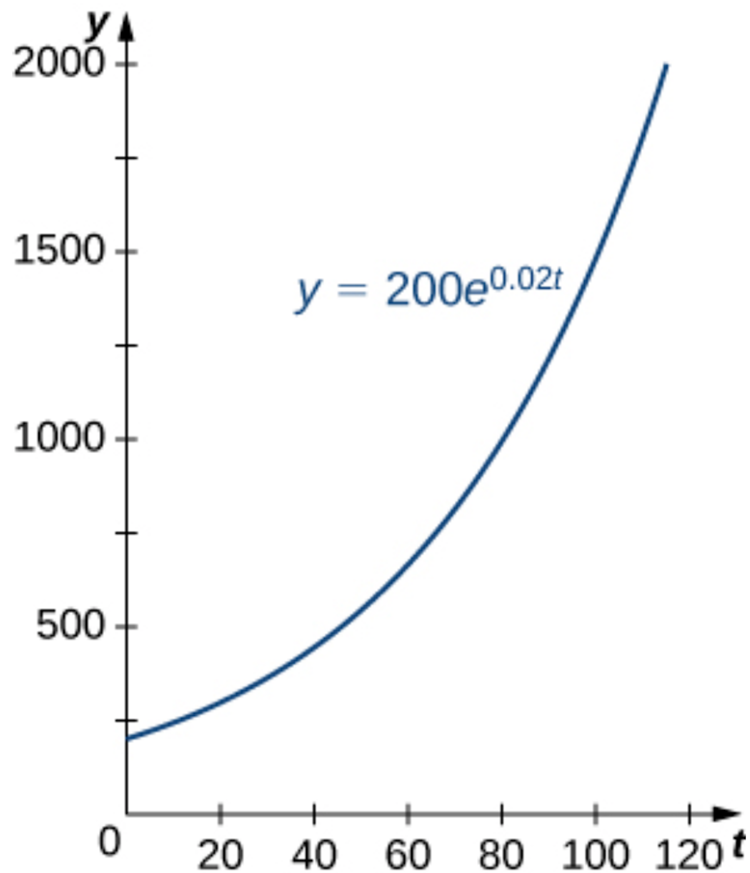


Figure 6.79 An example of exponential growth for bacteria.

Time (min)	Population Size (no. of bacteria)
10	244
20	298
30	364
40	445
50	544
60	664
70	811
80	991
90	1210
100	1478
110	1805
120	2205

Table 6.1 Exponential Growth of a Bacterial Population

(6.8) Exponential Growth and Decay

Example 6.42

Population Growth

Consider the population of bacteria described earlier. This population grows according to the function $f(t) = 200e^{0.02t}$, where t is measured in minutes. How many bacteria are present in the population after 5 hours (300 minutes)? When does the population reach 100,000 bacteria?

Solution

We have $f(t) = 200e^{0.02t}$. Then

$$f(300) = 200e^{0.02(300)} \approx 80{,}686.$$

There are 80,686 bacteria in the population after 5 hours.

To find when the population reaches 100,000 bacteria, we solve the equation

$$\begin{aligned} 100{,}000 &= 200e^{0.02t} \\ 500 &= e^{0.02t} \\ \ln 500 &= 0.02t \\ t &= \frac{\ln 500}{0.02} \approx 310.73. \end{aligned}$$

The population reaches 100,000 bacteria after 310.73 minutes.

(6.8) Exponential Growth and Decay

Let's now turn our attention to a financial application: compound interest. Interest that is not compounded is called *simple interest*. Simple interest is paid once, at the end of the specified time period (usually 1 year). So, if we put $\$1000$ in a savings account earning 2% simple interest per year, then at the end of the year we have

$$1000(1+0.02)=\$1020.$$

Compound interest is paid multiple times per year, depending on the compounding period. Therefore, if the bank compounds the interest every 6 months, it credits half of the year's interest to the account after 6 months. During the second half of the year, the account earns interest not only on the initial $\$1000,$ but also on the interest earned during the first half of the year. Mathematically speaking, at the end of the year, we have

$$1000\left(1+\frac{0.02}{2}\right)^2=\$1020.10.$$

(6.8) Exponential Growth and Decay

Similarly, if the interest is compounded every 4 months, we have

$$1000\left(1 + \frac{0.02}{3}\right)^3 = \$1020.13,$$

and if the interest is compounded daily (365 times per year), we have \$1020.20. If we extend this concept, so that the interest is compounded continuously, after t years we have

$$1000 \lim_{n \to \infty}\left(1 + \frac{0.02}{n}\right)^{nt}.$$

(6.8) Exponential Growth and Decay

Now let's manipulate this expression so that we have an exponential growth function. Recall that the number e can be expressed as a limit:

$$e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m.$$

Based on this, we want the expression inside the parentheses to have the form $(1 + 1/m)$. Let $n = 0.02m$. Note that as $n \to \infty$, $m \to \infty$ as well. Then we get

$$1000 \lim_{n \to \infty} \left(1 + \frac{0.02}{n}\right)^{nt} = 1000 \lim_{m \to \infty} \left(1 + \frac{0.02}{0.02m}\right)^{0.02mt} = 1000 \left[\lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m\right]^{0.02t}.$$

We recognize the limit inside the brackets as the number e. So, the balance in our bank account after t years is given by $1000e^{0.02t}$. Generalizing this concept, we see that if a bank account with an initial balance of $\$P$ earns interest at a rate of $r\%$, compounded continuously, then the balance of the account after t years is

$$\text{Balance} = Pe^{rt}.$$

(6.8) Exponential Growth and Decay

Example 6.43

Compound Interest

A 25-year-old student is offered an opportunity to invest some money in a retirement account that pays 5% annual interest compounded continuously. How much does the student need to invest today to have $\$1$ million when she retires at age 65? What if she could earn 6% annual interest compounded continuously instead?

Solution

We have

$$\begin{aligned} 1{,}000{,}000 &= Pe^{0.05(40)} \\ P &= 135{,}335.28. \end{aligned}$$

She must invest $\$135{,}335.28$ at 5% interest.

If, instead, she is able to earn 6%, then the equation becomes

$$\begin{aligned} 1{,}000{,}000 &= Pe^{0.06(40)} \\ P &= 90{,}717.95. \end{aligned}$$

In this case, she needs to invest only $\$90{,}717.95$. This is roughly two-thirds the amount she needs to invest at 5%. The fact that the interest is compounded continuously greatly magnifies the effect of the 1% increase in interest rate.

(6.8) Exponential Growth and Decay

If a quantity grows exponentially, the time it takes for the quantity to double remains constant. In other words, it takes the same amount of time for a population of bacteria to grow from 100 to 200 bacteria as it does to grow from 10,000 to 20,000 bacteria. This time is called the doubling time. To calculate the doubling time, we want to know when the quantity reaches twice its original size. So we have

$$\begin{aligned} 2y_0 &= y_0 e^{kt} \\ 2 &= e^{kt} \\ \ln 2 &= kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

(6.8) Exponential Growth and Decay

Definition

If a quantity grows exponentially, the **doubling time** is the amount of time it takes the quantity to double. It is given by

$$\text{Doubling time } = \frac{\ln 2}{k}.$$

(6.8) Exponential Growth and Decay

Example 6.44

Using the Doubling Time

Assume a population of fish grows exponentially. A pond is stocked initially with 500 fish. After 6 months, there are 1000 fish in the pond. The owner will allow his friends and neighbors to fish on his pond after the fish population reaches $10{,}000$. When will the owner's friends be allowed to fish?

Solution

We know it takes the population of fish 6 months to double in size. So, if t represents time in months, by the doubling-time formula, we have $6 = (\ln 2)/k$. Then, $k = (\ln 2)/6$. Thus, the population is given by $y = 500e^{((\ln 2)/6)t}$. To figure out when the population reaches $10{,}000$ fish, we must solve the following equation:

$$\begin{aligned} 10{,}000 &= 500e^{(\ln 2/6)t} \\ 20 &= e^{(\ln 2/6)t} \\ \ln 20 &= \left(\frac{\ln 2}{6}\right)t \\ t &= \frac{6(\ln 20)}{\ln 2} \approx 25.93. \end{aligned}$$

The owner's friends have to wait 25.93 months (a little more than 2 years) to fish in the pond.

(6.8) Exponential Growth and Decay

Exponential Decay Model

Rule: Exponential Decay Model

Systems that exhibit **exponential decay** behave according to the model

$$y = y_0 e^{-kt},$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the *decay constant*.

(6.8) Exponential Growth and Decay

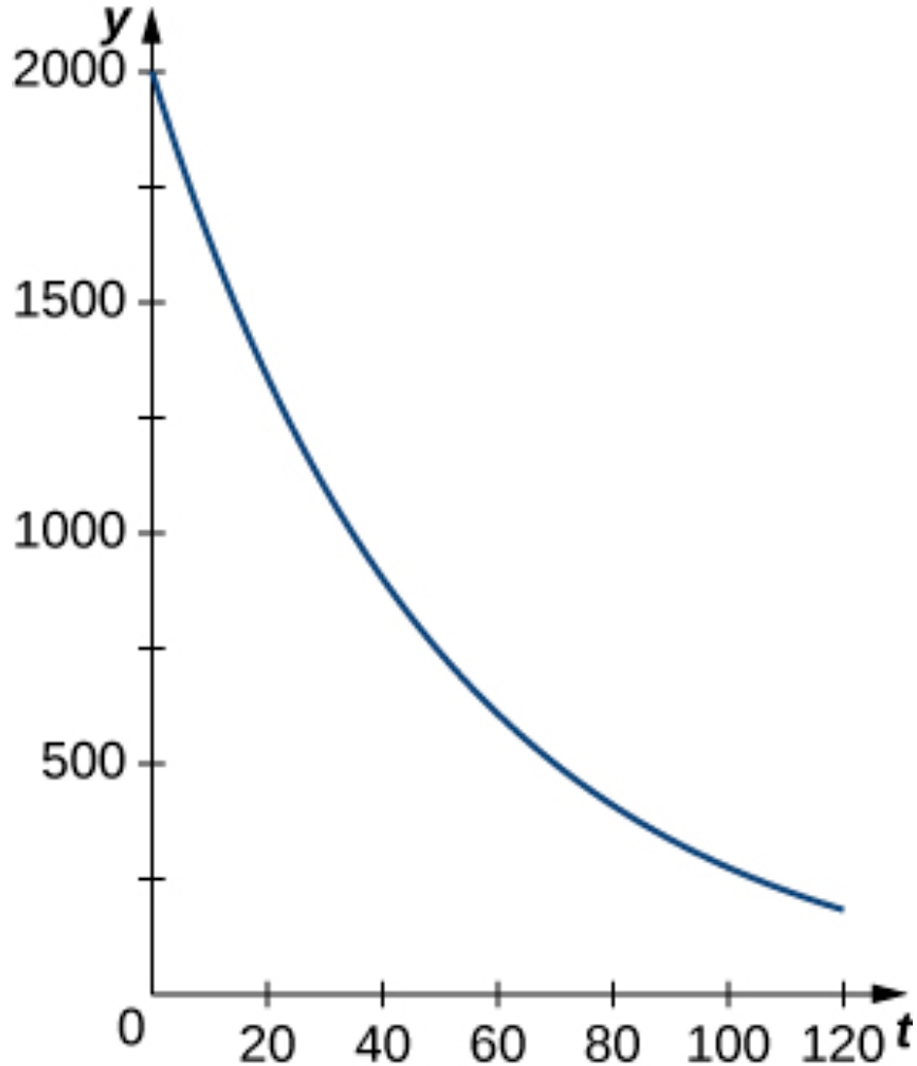


Figure 6.80 An example of exponential decay.

(6.8) Exponential Growth and Decay

Let's look at a physical application of exponential decay. Newton's law of cooling says that an object cools at a rate proportional to the difference between the temperature of the object and the temperature of the surroundings. In other words, if T represents the temperature of the object and T_a represents the ambient temperature in a room, then

$$T' = -k(T - T_a).$$

Note that this is not quite the right model for exponential decay. We want the derivative to be proportional to the function, and this expression has the additional T_a term. Fortunately, we can make a change of variables that resolves this issue. Let $y(t) = T(t) - T_a$. Then $y'(t) = T'(t) - 0 = T'(t)$, and our equation becomes

$$y' = -ky.$$

(6.8) Exponential Growth and Decay

From our previous work, we know this relationship between y and its derivative leads to exponential decay. Thus,

$$y = y_0 e^{-kt},$$

and we see that

$$\begin{aligned} T - T_a &= (T_0 - T_a)e^{-kt} \\ T &= (T_0 - T_a)e^{-kt} + T_a \end{aligned}$$

where T_0 represents the initial temperature. Let's apply this formula in the following example.

(6.8) Exponential Growth and Decay

Example 6.45

Newton's Law of Cooling

According to experienced baristas, the optimal temperature to serve coffee is between $155°F$ and $175°F$. Suppose coffee is poured at a temperature of $200°F$, and after 2 minutes in a $70°F$ room it has cooled to $180°F$. When is the coffee first cool enough to serve? When is the coffee too cold to serve? Round answers to the nearest half minute.

(6.8) Exponential Growth and Decay

Solution

We have

$$
\begin{aligned}
T &= (T_0 - T_a)e^{-kt} + T_a \\
180 &= (200 - 70)e^{-k(2)} + 70 \\
110 &= 130e^{-2k} \\
\frac{11}{13} &= e^{-2k} \\
\ln\frac{11}{13} &= -2k \\
\ln 11 - \ln 13 &= -2k \\
k &= \frac{\ln 13 - \ln 11}{2}.
\end{aligned}
$$

Then, the model is

$$T = 130e^{(\ln 11 - \ln 13/2)t} + 70.$$

(6.8) Exponential Growth and Decay

The coffee reaches $175°\text{F}$ when

$$\begin{aligned}
175 &= 130e^{(\ln 11 - \ln 13/2)t} + 70 \\
105 &= 130e^{(\ln 11 - \ln 13/2)t} \\
\frac{21}{26} &= e^{(\ln 11 - \ln 13/2)t} \\
\ln \frac{21}{26} &= \frac{\ln 11 - \ln 13}{2} t \\
\ln 21 - \ln 26 &= \frac{\ln 11 - \ln 13}{2} t \\
t &= \frac{2(\ln 21 - \ln 26)}{\ln 11 - \ln 13} \approx 2.56.
\end{aligned}$$

(6.8) Exponential Growth and Decay

The coffee can be served about 2.5 minutes after it is poured. The coffee reaches 155°F at

$$\begin{aligned} 155 &= 130e^{(\ln 11 - \ln 13/2)t} + 70 \\ 85 &= 130e^{(\ln 11 - \ln 13)t} \\ \frac{17}{26} &= e^{(\ln 11 - \ln 13)t} \\ \ln 17 - \ln 26 &= \left(\frac{\ln 11 - \ln 13}{2}\right)t \\ t &= \frac{2(\ln 17 - \ln 26)}{\ln 11 - \ln 13} \approx 5.09. \end{aligned}$$

The coffee is too cold to be served about 5 minutes after it is poured.

(6.8) Exponential Growth and Decay

Just as systems exhibiting exponential growth have a constant doubling time, systems exhibiting exponential decay have a constant half-life. To calculate the half-life, we want to know when the quantity reaches half its original size. Therefore, we have

$$\begin{aligned} \frac{y_0}{2} &= y_0 e^{-kt} \\ \frac{1}{2} &= e^{-kt} \\ -\ln 2 &= -kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

Note: This is the same expression we came up with for doubling time.

(6.8) Exponential Growth and Decay

Definition

If a quantity decays exponentially, the **half-life** is the amount of time it takes the quantity to be reduced by half. It is given by

$$\text{Half-life} = \frac{\ln 2}{k}.$$

(6.8) Exponential Growth and Decay

Example 6.46

Radiocarbon Dating

One of the most common applications of an exponential decay model is carbon dating. Carbon-14 decays (emits a radioactive particle) at a regular and consistent exponential rate. Therefore, if we know how much carbon was originally present in an object and how much carbon remains, we can determine the age of the object. The half-life of carbon-14 is approximately 5730 years—meaning, after that many years, half the material has converted from the original carbon-14 to the new nonradioactive nitrogen-14. If we have 100 g carbon-14 today, how much is left in 50 years? If an artifact that originally contained 100 g of carbon now contains 10 g of carbon, how old is it? Round the answer to the nearest hundred years.

(6.8) Exponential Growth and Decay

Solution

We have

$$\begin{aligned} 5730 &= \frac{\ln 2}{k} \\ k &= \frac{\ln 2}{5730}. \end{aligned}$$

So, the model says

$$y = 100e^{-(\ln 2/5730)t}.$$

In 50 years, we have

$$\begin{aligned} y &= 100e^{-(\ln 2/5730)(50)} \\ &\approx 99.40. \end{aligned}$$

Therefore, in 50 years, 99.40 g of carbon-14 remains.

(6.8) Exponential Growth and Decay

To determine the age of the artifact, we must solve

$$\begin{aligned} 10 &= 100e^{-(\ln 2/5730)t} \\ \frac{1}{10} &= e^{-(\ln 2/5730)t} \\ t &\approx 19035. \end{aligned}$$

The artifact is about 19,000 years old.