Chapter 3 Derivatives

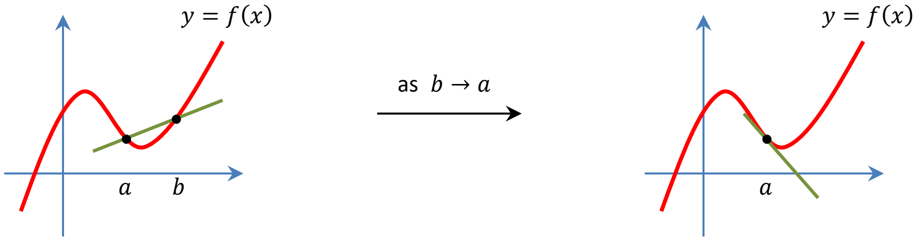
1. Differentiability and derivatives

Remark 3.1 Let f be a function defined on an interval and let $a \neq b$ be distinct real numbers in that interval. Let's consider the quantity

$$\frac{f(b)-f(a)}{b-a}.$$

- In geometry, $\frac{f(b)-f(a)}{b-a}$ is the **slope of the "secant line"** joining the two points $(a, f(a))$ and $(b, f(b))$ on the graph of f.
- In physics, if $a < b$, then $\frac{f(b)-f(a)}{b-a}$ is the **average rate of change** of f from a to b.
- In kinematics, if $f(t)$ represents the position of a moving particle after t seconds from an initial time and if $a < b$, then $\frac{f(b)-f(a)}{b-a}$ is the **average velocity** of the particle from the a^{th} second to the b^{th} second.

We aim to study the limit of the above quantity when a and b becomes so close to each other that the line joining the points $(a, f(a))$ and $(b, f(b))$ on the graph of f (a ***secant line***) becomes a line which touches the graph of f at the point $(a, f(a))$ (a ***tangent line***).



In general, such a limit may or may not exist. So we make the following definition.

Definition 3.2 Let a be a real number and f be a function. f is said to be ***differentiable at*** $\boldsymbol{a}$ if the limit

$$\lim_{b\to a}\frac{f(b)-f(a)}{b-a},$$

or equivalently the limit

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

(obtained with a change of variable $h = b - a$), exists as a finite real number. f is said to be ***differentiable on an interval*** if it is differentiable at every number in that interval.

Definition 3.3 Let f be a function.

- Let a be a real number. If f is differentiable at a, then the ***derivative of f at a*** is defined by the limit
$$f'(a) := \lim_{b \to a} \frac{f(b) - f(a)}{b - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$
- Replacing the real number a by a real variable x, we say that the ***derivative of f*** is the function f' defined by
$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
for every x in the domain of f at which f is differentiable. The domain of this function f' is therefore a subset of the domain of f.
- The process of finding the derivative of a function is called ***differentiation***. So to ***differentiate*** the function f means to find its derivative f', i.e. to evaluate the above limit for each x in the domain of f.

Remark 3.4 Let a be a real number and let f be a function which is differentiable at a. Then the graph of f is smooth near a.

- In geometry, $f'(a)$ is the **slope of the tangent line** to the graph of f at the point $(a, f(a))$.
- In physics, $f'(a)$ is the **"instantaneous" rate of change** of f at a.
- In kinematics, if $f(t)$ represents the position of a moving particle after t seconds from an initial time, then $f'(a)$ is the **"instantaneous" velocity** of the particle at the a^{th} second.

Example 3.5 Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by
$$f(x) = x^2.$$
Find the derivative of f from definition.

Solution:

For every real number x, we have
$$\begin{aligned}\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x.\end{aligned}$$
Therefore f is differentiable everywhere on $\mathbb{R}$, and its derivative $f': \mathbb{R} \to \mathbb{R}$ is given by
$$f'(x) = 2x.$$